HANDBOOK

OF

PROBLEMS

IN

DIRECT FIRE.

BY

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PREFACE.

THIS book was prepared while the Author was engaged in teaching ballistics to student officers at the Artillery School, Fort Monroe; and most of the examples were selected from those which had been given out from time to time to the classes under his instruction, as exercises in the practical applications of the ballistic formulæ which the more advanced students were required to deduce. It was suggested to the Author, by officers of high rank, that a collection of these and similar examples, in book form, would be of permanent value, not only to the Artillery, but also to the other branches of the service, both regular and militia.

A very slight knowledge of mathematics is all that is required for the solution of the most important of the examples. It has come to the Author's knowledge that the first twelve problems have been taught successfully to non-commissioned officers whose whole stock of mathematics consisted of a little arithmetic and less algebra. In the later problems the symbol of integration has been introduced in a few instances—chiefly, however, for the sake of concise definitions. Wherever this symbol occurs it may be passed over without detriment to the practical applications.

It is believed that this is the first book of the kind ever published in any language; and the Author trusts that this will excuse whatever faults of arrangement or of execution may be detected by the reader. The solutions of the first seventeen problems are based upon the method first given to the world by captain (now Lieutenant-Colonel) Siacci, of the Italian

Artillery, in 1880, and which is now universally employed in Europe and America. In connection with Siacci's method the Author has here introduced the labor-saving, auxiliary equations of Captain Scipione (also of the Italian Artillery), and which he has rendered available, for the first time, to English and American Artillerists by extensive tables, prepared expressly for this work. The attentive reader will find scattered throughout the work methods and processes which he will seek for in vain elsewhere, and which it is hoped may be found useful to the practical Artillerist.

At the suggestion of the publishers an appendix (Appendix I) has been added, giving a concise, but quite complete, deduction of the formulas of Siacci's method, together with other matter which it is hoped may be acceptable to the mathematical reader. The Author has also added Appendix II on his own responsibility, giving the latest and best methods for the solution of problems in Mortar-firing.

Of the tables given in the book, those computed by the Author are the following: Table of Altitude Factors, page 89; Table of the values of B, page 153; Tables 2, 3, 4, and 5 in Problem XXI; Tables I, II, III, and V at the end of the book.

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PROBLEMS IN DIRECT FIRE.

INTRODUCTION.

It has been the object in the following pages to give practical solutions of all the important problems of gunnery relating to direct fire; and to illustrate the solutions by numerous examples, fully worked out in the manner which a considerable experience has shown to be the most simple and concise. For this purpose logarithms are habitually employed in making the numerical computations, as it is believed that by their use a considerable saving of time and labor is effected, and the liability to error reduced to a minimum. In the absence of a table of logarithms, however, nearly all the examples may be worked by simple arithmetic.

Five-place logarithms are sufficient for the correct solution of all gunnery problems; and four-place logarithms can often be used to advantage. An excellent five-place table has been compiled by Wentworth and Hill, and published by Ginn & Co. of Boston. This is the table that has been used in this work.

Definitions.—The following definitions of a few technical terms which will be constantly employed, are here given for convenient reference:

The *trajectory* is the curve described by the centre of gravity of the projectile. It is divided into the ascending and descending branches, the point of division being called the summit.

The *line of departure* is the prolongation of the axis of the

bore at the instant the projectile leaves the gun. It is therefore tangent to the trajectory at the muzzle.

The *angle of departure* is the angle which the line of departure makes with the horizontal plane.

The *angle of elevation* (or *depression*) is the angle which the axis of the bore, when the piece is laid, makes with the horizontal plane. It is sometimes called the quadrant elevation because it is often determined by applying the quadrant to the face of the piece.

Jump is the difference between the angle of elevation and angle of departure. It varies in value from an angle too small to be appreciable to one of a degree of arc or even more, according to the kind of carriage and platform employed. It also varies somewhat with the angle of elevation. It must be determined by experiment in each case.

Muzzle velocity is the velocity of the projectile on leaving the piece. It is sometimes called initial velocity or velocity of projection.

Remaining velocity is the velocity at any given point of the trajectory.

Final velocity is the velocity the projectile has in the descending branch when at the level of the gun.

The *range* is the horizontal distance from the muzzle of the gun to that point of the descending branch of the trajectory, called the point of fall, which is at the level of the gun. This term is also applied to the distance between the gun and the target, or between the gun and the point where the projectile strikes, whether above or below the level of the gun.

The *angle of fall* is the angle which the tangent to the trajectory at the point of fall makes with the horizontal plane passing through the muzzle.

Direct fire is with guns, with service charges, and angles of elevation not exceeding 15°.

Indirect (or *curved*) *fire* is with guns, howitzers, and mortars, with reduced charges (and therefore low velocities), and angles of elevation not exceeding 15°.

High-angle fire is when the angle of elevation exceeds 15°.

Notation.—The following notation will be employed:

g denotes acceleration of gravity, and will be taken at 32.16 f. s.

w, weight of a projectile in pounds.

d, diameter of a projectile in inches.

$\delta_{\prime}$, density of air two-thirds saturated with moisture, thermometer 60° (F.) and barometer 30 inches.

δ, density of air two-thirds saturated for *any* observed readings of the thermometer and barometer.

c, coefficient of reduction, depending upon the kind of projectile employed.

C, ballistic coefficient $= \dfrac{\delta_{\prime}}{\delta} \dfrac{w}{cd^2}$.

V, muzzle velocity in feet per second.

v, velocity at any point of the trajectory.

v_{ω}, velocity at point of fall.

v_0, velocity at summit of trajectory.

ϕ, angle of departure.

θ, angle which the tangent to the trajectory at any point makes with a horizontal plane; positive in the ascending and negative in the descending branch.

ω, angle of fall. This angle, which is really negative, will be regarded as positive.

x, y, rectangular co-ordinates of any point of the trajectory, the origin being at the muzzle of the gun, and the axis of x horizontal. To designate the co-ordinates of a particular point subscripts will be used as x_0, y_0, co-ordinates of the summit, etc.

t, time of describing any portion of the trajectory from the origin.

X, horizontal range.

T, time of flight from the origin to the point of fall.

u, an auxiliary quantity, function of the velocity and inclination, defined by the equation $u = v \dfrac{\cos\theta}{\cos\phi}$. u_0 will represent the value of u at the summit, and u_{ω} at the point of fall.

$S(V)$, $S(u)$, etc., space functions.

$A(V)$, $A(U)$, $A(u)$, etc., altitude functions.

$I(V)$, $I(U)$, $I(u)$, etc., inclination functions.
$T(V)$, $T(U)$, $T(u)$, etc., time functions.
$B(v)$, $M(v)$, drift functions.

E_0, energy of projectile at any point of its course $= \dfrac{wv^2}{4480g}$ foot-tons.

E_i, energy per inch of shot's circumference $= \dfrac{wv^2}{4480\pi dg}$ foot-tons $= \dfrac{E_0}{\pi d}$.

W, velocity of wind in feet per second.
W_p, velocity of wind parallel to range in feet per second.
W_n, velocity of wind normal to range in feet per second.
ρ, resistance of the air to a projectile's motion, in pounds.
τ, thickness of armor in inches.

General Formulæ.—The following fundamental equations give the values of x, t, y, and θ at any point of the trajectory, in terms of V, ϕ, u, and C. They are the basis of the solutions of most of the problems of direct fire. Their demonstration is given in Appendix 1.

$$x = C\{S(u) - S(V)\}, \qquad (1)$$

$$t = \frac{C}{\cos\phi}\left\{T(u) - T(V)\right\}, \qquad (2)$$

$$\frac{y}{x} = \tan\phi - \frac{C}{2\cos^2\phi}\left\{\frac{A(u) - A(V)}{S(u) - S(V)} - I(V)\right\}, \qquad (3)$$

$$\tan\theta = \tan\phi - \frac{C}{2\cos^2\phi}\left\{I(u) - I(V)\right\}, \qquad (4)$$

$$u = v\frac{\cos\theta}{\cos\phi}. \qquad (5)$$

Of these equations, (3) was first given by Major Siacci of the Italian Artillery; and the method of solving trajectories which is based upon it goes by the name of "Siacci's Method."

At the summit of the trajectory the motion of the projectile is horizontal; and therefore at this point $\theta = 0$. Substituting this value of θ in (4) and (5) and reducing, we have at the summit the following special equations which will be found of great use in the sequel:

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V), \quad \ldots \ldots \quad (6)$$

$$u_0 = \frac{v_0}{\cos \phi}. \quad \ldots \ldots \ldots \ldots \quad (7)$$

In these equations V (the muzzle velocity) and ϕ (the angle of departure) relate to the muzzle of the gun and may be considered constant. All the other symbols refer to any point of the trajectory; namely, x, y are the rectangular co-ordinates of any point, t is the time from the origin to the point (x, y), θ is the angle made by the tangent to the trajectory at the same point with a horizontal plane, and v is the corresponding velocity. u is an auxiliary quantity defined by (5). Equation (3) would be the equation of the trajectory were it possible to eliminate the variable u; but as this is impossible, the trajectory is determined by a combination of (1) and (3). The manner of using these and the following formulas will be fully explained under the appropriate problems.

Formulæ relating to the Horizontal Range.—By giving suitable values to the variables we may deduce a set of five equations for any point of the trajectory. For example, if we make $y = 0$ and $-\theta = \omega$, we shall have the equations for the point of fall; and the particular values of x, t, v, and u upon this supposition are designated by X, T, v_ω, u_ω. Equations (1), (2), (3), (4), and (5) then become, respectively,

$$X = C\{S(u_\omega) - S(V)\}, \quad \ldots \ldots \ldots \ldots \quad (8)$$

$$T = \frac{C}{\cos \phi}\left\{ T(u_\omega) - T(V) \right\}, \quad \ldots \ldots \quad (9)$$

$$\sin 2\phi = C\left\{\frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)} - I(V)\right\}, \quad . \quad . \quad . \quad (10)$$

$$\tan \omega = \frac{C}{2\cos^2\phi}\left\{I(u_\omega) - I(V)\right\} - \tan\phi, \quad . \quad (11)$$

$$u_\omega = v_\omega \frac{\cos \omega}{\cos \phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

We may give to (11) a form better adapted for computation by combining it with (10), eliminating $I(V)$; thus,

$$\tan \omega = \frac{C}{2\cos^2\phi}\left\{I(u_\omega) - \frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)}\right\}. \quad . \quad (13)$$

It is evident from (6) and (10) that we have the relation

$$I(u_0) = \frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and therefore (10) and (13) may be written, with great economy of space,

$$\sin 2\phi = C\{I(u_0) - I(V)\}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\tan \omega = \frac{C}{2\cos^2\phi}\left\{I(u_\omega) - I(u_0)\right\}. \quad . \quad . \quad (16)$$

When ϕ and ω are both small we have, approximately,

$$2\cos^2\phi \tan\omega = 2\cos^2\omega \tan\omega = \sin 2\omega;$$

and upon this hypothesis (16) becomes

$$\sin 2\omega = C\{I(u_\omega) - I(u_0)\}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Auxiliary Formulæ.—The above formulæ are sufficient in connection with a table of the S, A, I, and T functions, for the solution of all problems of direct fire. But some of these solutions are indirect and tentative, and therefore very laborious. We may, however, by the help of certain auxiliary formulæ due to Captain Braccialini Scipione of the Italian Artillery, in con-

nection with suitable tables computed by the Author of this work, reduce the numerical labor very materially. These formulæ we now proceed to deduce.

Let us suppose that we have given the ballistic coefficient (C), the muzzle velocity (V), and the range (X); and that we wish to compute the angle of departure (ϕ), the angle of fall (ω), and the time of flight (T). To do this we would first compute u_ω by the equation

$$S(u_\omega) = \frac{X}{C} + S(V)$$

derived from (8); and then ϕ, ω, and T by (10), (13), and (9), respectively.

From this it is evident that ϕ, ω, and T are, for given values of V, functions of $\frac{X}{C}$, and conversely. It is also apparent that the numerical labor of calculation would be greatly shortened if the quantities

$$\frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)} - I(V)$$

and

$$I(u_\omega) - \frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)}$$

could be taken directly from a table with the arguments $\frac{X}{C}$ and V. Let us put, then,

$$\frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)} - I(V) = A$$

and

$$I(u_\omega) - \frac{A(u_\omega) - A(V)}{S(u_\omega) - S(V)} = B.$$

We shall then have from (10) and (13)

$$\sin 2\phi = AC, \quad \ldots \ldots \ldots \quad (18)$$

$$\tan \omega = \frac{BC}{2 \cos^2 \phi}, \quad \ldots \ldots \quad (19)$$

and, for small angles (not exceeding about 5°), from (17),

$$\sin 2\omega = BC. \quad \ldots \ldots \ldots \quad (20)$$

We may also simplify the general formulæ (3) and (4) by similar means. Thus let

$$\frac{A(u) - A(V)}{S(u) - S(V)} - I(V) = a,$$

$$I(u) - \frac{A(u) - A(V)}{S(u) - S(V)} = b,$$

$$I(u) - I(V) = a + b = m.$$

We then have from (1), (3), and (4),

$$S(u) = \frac{x}{C} + S(V), \quad \ldots \ldots \ldots \quad (21)$$

$$y = x \tan \phi \left\{ 1 - \frac{aC}{\sin 2\phi} \right\}, \quad \ldots \quad (22)$$

and

$$\tan \theta = \tan \phi \left\{ 1 - \frac{mC}{\sin 2\phi} \right\}, \quad \ldots \ldots \quad (23)$$

Let the distinction between A, B, and a, b, m, be clearly noted: The capital letters refer to $\frac{X}{C}$, where X is the range; while the small letters refer to $\frac{x}{C}$, where x is the abscissa of any point whatever of the trajectory. a and b are however taken from the same tables, respectively, as A and B, though not referring necessarily to the same things.

Substituting in (22) and (23) for $\sin 2\phi$ its value from (18), they become

$$y = x \tan \phi \left(1 - \frac{a}{A}\right) \quad . \quad . \quad . \quad . \quad (24)$$

and

$$\tan \theta = \tan \phi \left(1 - \frac{m}{A}\right). \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If in (25) we make $\theta = 0$, we deduce for the summit of the trajectory the identity

$$m = A$$

which has already been established. (See Eq. 14.)

Ballistic Coefficient.—The ballistic coefficient (C) which appears in nearly all the preceding formulæ depends upon the weight, diameter, and smoothness of the projectile, its steadiness in flight, and the density of the air it encounters. Its expression is

$$C = \frac{\delta_{\prime}}{\delta} \frac{w}{cd^2},$$

in which d is the diameter of the projectile in inches, w its weight in pounds, c a factor, called the "coefficient of reduction," depending upon the kind of projectile used, its steadiness, etc., and $\frac{\delta_{\prime}}{\delta}$ a factor depending upon the density of the air.

Coefficient of Reduction.—For guns and projectiles similar in every respect to those used in making the experiments upon which the tables are based (and which may be called the *standard*), the coefficient of reduction will be unity. If the qualities of the gun and projectile are such that the latter meets with a greater resistance than the standard of the same diameter, the effect is the same as though the area exposed to resistance (that is, the area of its greatest cross-section) were increased; and therefore in this case c must be greater than unity. On the other hand, if the resistance to a projectile be less than the standard, c will be less than unity. The method

for determining the value of c in any case, by experiment, will be given farther on. For our new breech-loading guns and the Krupp guns we may assume $c = 0.9$ without much error.

Density of the Air.—In the factor $\frac{\delta_{\prime}}{\delta}$, $\delta_{\prime}$ is the standard density of the air to which the experiments upon which the tables are based were reduced; and δ is the observed density at the time of firing. The value of this factor for any observed temperature (Fahrenheit) and barometric pressure may be taken from Table III. In computing this table the air was supposed to be two-thirds saturated with moisture, which is not far from the case on our sea-coast; and therefore the hygrometric condition of the atmosphere need not be noticed, the only observations necessary being those of the thermometer and barometer.

Example. At target practice the thermometer stood at 88°.5 and barometer at 30.194 in. What was the value of $\frac{\delta_{\prime}}{\delta}$?

From Table III we see at a glance that for a temperature of 88°.5 the value $\frac{\delta_{\prime}}{\delta}$ for 30 inches is 1.056, and for 31 inches it is 1.022; while the difference between them is 0.034. Therefore

$$\frac{\delta_{\prime}}{\delta} = 1.056 - 0.194 \times 0.034 = 1.049.$$

Ballistic Tables.—Table I gives the values of the functions $S(v)$, $A(v)$, $I(v)$, and $T(v)$; and, also, the drift functions $B(v)$ and $M(v)$, which will be described farther on; and extends from $v = 2800$ f. s. to $v = 400$ f. s. It is based upon a discussion of the values of Bashforth's coefficients K, which he determined from his experiments at Shoeburyness, between the years 1865 and 1880. The table was computed in 1884, and first appeared in the second Artillery School edition of the Author's work on Exterior Ballistics, January, 1885.

For convenience of interpolation the first differences are

given in adjacent columns; and as the second differences rarely exceed eight units of the last order, it will hardly ever be necessary to consider them in using this table.

Table II, the ballistic table for spherical projectiles, is based upon the experiments made by General Mayevski at St. Petersburg, in 1868; and extends from $v = 2000$ to $v = 450$. It was computed in 1883, and is the only ballistic table for spherical projectiles, based upon Siacci's method, yet published.

Formulæ for Interpolation.—To find the values of $S(v)$, $A(v)$, etc., when v lies between two consecutive values of v as given in Tables I and II, and when second differences are taken into account, we proceed as follows:

Let v_0 and v_1 be the two consecutive values of the argument between which v lies. Let $v_0 - v_1 = h$; and designate the first and second differences of the function under consideration, by Δ_1 and Δ_2. Then if we symbolize the function by $f(v)$ we shall have, since $f(v)$ increases while v decreases,

$$f(v) = f(v_0) + \frac{v_0 - v}{h}\Delta_1 - \frac{v_0 - v}{h}\left(1 - \frac{v_0 - v}{h}\right)\frac{\Delta_2}{2},$$

by means of which $f(v)$ can be computed.

Conversely, if $f(v)$ is given and our object is to find v, we have

$$\frac{v_0 - v}{h}\Delta_1 = f(v) - f(v)_0 + \frac{v_0 - v}{h}\left(1 - \frac{v_0 - v}{h}\right)\frac{\Delta_2}{2}.$$

In using this last formula, first compute $\frac{v_0 - v}{h}$ by omitting the second term of the second member (which is usually very small), and then supply this term, using the approximate value of $\frac{v_0 - v}{h}$ already found.

If the second differences are too small to be taken into account, that is, less than eight units of the last order, the above formulæ become, respectively,

$$f(v) = f(v_0) + \frac{v_0 - v}{h}\Delta_1$$

and

$$v = v_0 - \frac{h}{\Delta_1}\Big(f(v) - f(v_0)\Big),$$

which express the well-known rules of proportional parts.

In Table I the values of h are as follows:

From $v = 2800$ to $v = 2200$, $h = 50$;
" $v = 2200$ " $v = 1600$, $h = 10$;
" $v = 1600$ " $v = 1320$, $h = 5$;
" $v = 1320$ " $v = 1160$, $h = 2$;
" $v = 1160$ " $v = 400$, $h = 1$.

And in Table II,

From $v = 2000$ to $v = 1200$, $h = 10$;
" $v = 1200$ " $v = 450$, $h = 5$.

Example 1. Find $S(v)$ from Table I, when $v = 1847.6$.

We have $v_0 = 1850$, $f(v_0) = S(v_0) = 2916.9$, $h = 10$, and $\Delta_1 = 38.2$.

$$\therefore\ S(v) = 2916.9 + \frac{1850 - 1847.6}{10} \times 38.2 = 2926.0.$$

Example 2. Find from Table II the value of $A(v)$ when $v = 1023.7$.

We have $v_0 = 1025$, $f(v_0) = A(v_0) = 159.15$, $\Delta_1 = 3.84$, $h = 5$, and $v_0 - v = 1.3$.

$$\therefore\ A(1023.7) = 159.15 + \frac{1.3 \times 3.84}{5} = 160.15.$$

Example 3. Suppose for an oblong projectile we have found $S(v) = 12870.2$. What is the value of v?

We find in Table I the first value of $S(v)$ less than 12870.2 to be 12856.7, which corresponds to $v_0 = 719$. We also find $\Delta_1 = 23.6$ and $h = 1$.

$$\therefore\ v = 719 - \frac{12870.2 - 12856.7}{23.6}$$

$$= 719 - 0.57 = 718.43.$$

Example 4. What is $A(562.7)$ by Table II?

We have $v_0 = 565$, $A(565) = 2620.0$, $h = 5$, $\Delta_1 = 104.3$, and $\Delta_2 = 4.8$.

$$\therefore A(562.7) = 2620.0 + \frac{2.3}{5} \times 104.3 - \frac{2.3}{5} \times \frac{2.7}{5} \times 4.8$$
$$= 2620.0 + 48.0 - 1.2 = 2666.8.$$

Auxiliary Tables.—We have computed three auxiliary tables, for oblong projectiles only, which give the values of the auxiliary quantities A, B, and m, respectively. They are based upon Table I, and are to be used in connection with it. These tables have each two arguments, viz., $\frac{X}{C}$ (called z), found in the first vertical column, and the velocity, V, in the upper horizontal column. They give the values of the functions to four decimals for equidistant values of z from 100 to 7000, and for V from 1200 f. s. to 2250 f. s. The constant difference between the values of z is 100, and that between V is 50. In the columns Δ_z are given the differences between the consecutive values of the functions relative to the same value of V and corresponding to an increase of 100 in the value of z; and in the columns Δ_v are given the corresponding differences for an increase of 50 in the value of V. In the interpolation formulæ these differences are to be used as positive whole numbers.

If z and V are given values of the arguments intermediate to those found in the tables and we wish for the corresponding values of the function [symbolized as $f(z, V)$], we have

$$f(z, V) = f(z_0, V_0) + \frac{z - z_0}{100}\Delta_z - \frac{V - V_0}{50}\Delta_v,$$

in which z_0 and V_0 are the next smaller tabulated values of the arguments to those given, and $f(z_0, V_0)$ is the corresponding tabular value of the function. If $f(z, V)$ and V are given to find z, we have from the above equation

$$z = z_0 + \frac{100}{\Delta_z}\left\{\frac{V - V_0}{50}\Delta_v + f(z, V) - f(z_0, V_0)\right\}.$$

Similarly, if $f(z, V)$ and z are given to find V, we have

$$V = V_0 + \frac{50}{\Delta_v}\left\{\frac{z - z_0}{100}\Delta_z + f(z_0, V_0) - f(z, V)\right\}.$$

Example 1. What is the value of A when $z = 5279.3$ and $V = 1623.4$?

We have $z_0 = 5200$, $V_0 = 1600$, $f(z_0, V_0) = 0.1089$, $\Delta_z = 30$, and $\Delta_v = 55$.

$$\therefore f(z, V) = A = 0.1089 + \frac{79.3}{100} \times 30 - \frac{23.4}{50} \times 55;$$

$$\therefore A = 0.1087.$$

Example 2. Given $B = 0.1430$ and $V = 1740$, to find z.

We have $V_0 = 1700$; and running down the column we see that $z_0 = 5200$ and $f(z_0, V_0) = 0.1455$. Also, $\Delta_z = 47$ and $\Delta_v = 55$.

$$\therefore z = 5200 + \frac{100}{47}\left\{\frac{40}{50} \times 55 + 1430 - 1455\right\} = 5240.$$

Example 3. Given $m = 0.2400$ and $z = 5250$, to find V.

We have $z_0 = 5200$, $V_0 = 1700$, $f(z_0, V_0) = 0.2236$, $\Delta_z = 75$, and $\Delta_v = 105$.

$$\therefore V = 1700 + \frac{50}{105}\left\{\frac{50}{100} \times 75 + 2436 - 2400\right\} = 1735 \text{ f. s.}$$

It sometimes happens that the given value of z, or of V, is found in the table. In this case the interpolation is shortened since we have $z - z_0$, or $V - V_0$, as the case may be, equal to zero.

Example 4. What is the value of A when $z = 3947.3$ and $V = 1400$ f. s.?

We have $z_0 = 3900$, $\Delta_z = 31$, $z - z_0 = 47.3$, $f(z_0, V_0) = 0.0916$, and $V_0 = V = 1400$ f. s.

$$\therefore A = 0.0916 + .0015 = 0.0931.$$

Example 5. Given $m = 0.1834$ and $z = 4300$, to find V.

We have $z_0 = z = 4300$, $V_0 = 1650$, $f(z_0, V_0) = 0.1900$, and $\Delta_z = 91$.

$$\therefore\ V_0 = 1650 + \frac{50}{91}\left\{ 1900 - 1834 \right\}$$

$$= 1650 + 36 = 1686 \text{ f. s.}$$

Problem I.

Given the muzzle velocity (V), *and data for the ballistic coefficient* (C), *to calculate the velocity* (v) *at a distance* (x) *from the gun.*

Solution. First compute C by the formula

$$C = \frac{\delta_{\prime}}{\delta} \frac{w}{cd^2},$$

taking the value of $\frac{\delta_{\prime}}{\delta}$ from Table III, for the given temperature and barometric pressure, and using the proper value of c for the gun and projectile under consideration. If no observation of the state of the atmosphere has been made, and if, besides, the value of c is not known, the best that can be done will be to compute C by the equation

$$C = \frac{w}{d^2}.$$

For all smooth-bore guns and for the older rifled guns in our service $c = 1$. For the new B. L. rifles $c = 0.9$ approximately.

Next, take from the proper table (Table I, for elongated, and Table II, for spherical projectiles), the function $S(V)$ for the given muzzle velocity V. Then compute $S(u)$ by the equation (derived from Eq. (1)),

$$S(u) = z + S(V),$$

in which

$$z = \frac{x}{C},$$

and take the corresponding value of u from the table. Then from (5) we have

$$v = u\frac{\cos\phi}{\cos\theta};$$

in which v, u, and θ refer to the point of the trajectory whose abscissa is x.

If the angle of departure is small, not exceeding 10°, θ will also be small and the ratio of the cosines will be nearly unity. Under these conditions we may for practical purposes assume that

$$v = u.$$

This assumption, namely, that

$$\frac{\cos\phi}{\cos\theta} = 1,$$

implies that the motion of the projectile is horizontal; that is, that the trajectory is a horizontal right line; and, therefore, that gravity, which acts vertically, neither increases nor retards the projectile's motion. Should the trajectory be so curved that the ratio of the cosines differs materially from unity, it will be necessary to know the values of ϕ and θ in order to compute v with the greatest accuracy, as will be exemplified in subsequent problems.

Example 1. Calculate the final velocity of a 15-inch solid shot for a range of 1000 yards, the muzzle velocity being 1700 f. s. and atmosphere normal.

We have given $V = 1700$, $d = 14.87$, $w = 450$, and $X = 3000$, to find v_ω.

NOTE.—Hereafter we shall omit the subscript ω as being unnecessary, since the nature of the problem will determine what velocity is meant.

We have the formulas

$$C = \frac{w}{d^2},$$

$$z = \frac{X}{C},$$

$$S(v) = z + S(V),$$

to compute v. The work by logarithms may be concisely and conveniently arranged as follows:

$$\begin{array}{rl}
\log w &= 2.65321 \\
2 \log d &= \underline{2.34462} \\
\log C &= 0.30859 \\
\log X &= \underline{3.47712} \\
\log 1474 &= 3.16853 = \log z. \\
\text{(Table II)} \quad S(V) &= \underline{\ \ 798} \\
S(v) &= 2272 \qquad \therefore v = 1259 \text{ f. s.}
\end{array}$$

By eliminating C from the above equations, the expression for v may be written

$$S(v) = \frac{d^2}{w} X + S(V).$$

Employing this last formula, the computation by logarithms is as follows:

$$\begin{array}{rl}
2 \log d &= 2.34462 \\
\log X &= 3.47712 \\
\text{a. c. } \log w &= \underline{7.34679} \\
\log 1474 &= 3.16853 \\
S(V) &= \underline{\ \ 798} \\
S(v) &= 2272 \qquad \therefore v = 1259 \text{ f. s.}
\end{array}$$

This last solution involves the writing of fewer figures than the first; and is as short as it is possible to make it. Generally, however, in ballistic problems, the value of C must be determined before the problem can be solved; and then the first arrangement is preferable.

We may also write the expression for v as follows:

$$S(v) = \frac{(14.87)^2 \times 3000}{450} + 798,$$

and perform the arithmetical operations indicated.

Example 2. Calculate the final velocity of an 8-inch elongated projectile fired from the B. L. rifle, with a muzzle velocity the same as in Ex. 1, for a range of 3500 yards.

Here $d = 8$; $c = 0.9$ (and, therefore, $cd^2 = 57.6$); $V = 1700$; $w = 290$; $X = 10500$; and $\frac{\delta_{\prime}}{\delta} = 1$.

We have, therefore,

$$\begin{aligned} \log cd^2 &= 1.76042 \\ \log X &= 4.02119 \\ \text{a. c. } \log w &= 7.53760 \\ \hline \log 2085.5 &= 3.31921 \end{aligned}$$

(Table I) $\quad S(V) = 3512.1$

$$S(v) = 5597.6 \qquad \therefore v = 1264.5 \text{ f. s.}$$

These two examples show that with the same muzzle velocity the lighter elongated projectile has about the same final velocity for a range of 3500 yards as the heavier spherical projectile has for 1000 yards. They illustrate very strikingly the superiority of rifled guns over smooth-bores, in carrying destructive energy to the object to be destroyed.

Example 3. With a charge of 8 pounds sphero-hexagonal powder the 4.5-inch siege-gun gives a muzzle velocity of 1400 f. s., weight of projectile 35 pounds. What is the final velocity for a range of 1516 yards? thermometer 82°, barometer 29.75 inches.

NOTE.—For this gun $\mathfrak{c} = 1$.

We have given $V = 1400$, $d = 4.5$, $w = 35$, $X = 4548$, and $\frac{\delta_{\prime}}{\delta} = 1.052$.

$$\begin{aligned}
\log \frac{\delta_{\prime}}{\delta} &= 0.02202 \\
\log w &= 1.54407 \\
\text{a.c.} \log d^2 &= 8.69357 \\
\hline
\log C &= 0.25966 \\
\log X &= 3.65782 \\
\hline
\log 2501.3 &= 3.39816 \\
S(V) &= 4878.6 \\
\hline
S(v) &= 7379.9 \qquad \therefore v = 1028 \text{ f. s.}
\end{aligned}$$

Example 4. What would be the velocity in the above example at the middle point of the range?

In this case we have $x = 2274$ ft. and the remaining data the same as above.

$$\begin{aligned}
\log x &= 3.35679 \\
\log C &= 0.25966 \\
\hline
\log 1250.6 &= 3.09713 \\
S(V) &= 4878.6 \\
\hline
S(v) &= 6129.2 \qquad \therefore v = 1178 \text{ f. s.}
\end{aligned}$$

Comparison of Computed with Measured Velocities.— This problem is useful for testing the accuracy of the ballistic tables by comparing the computed velocity of a projectile at a considerable distance from the gun, with the velocity measured at the same point with a chronograph.

The only velocities, measured at a distance from the gun, known at the Artillery School, are those taken at Meppen, and published from time to time by Krupp, in his valuable *Expériences de tir;* and those executed by the Hotchkiss Ordnance Company at Gâvre. In both these the metric system of units is employed; and to reduce all the given data to English units would involve considerable labor even with the help of the "Tables of Reduction" found at the end of this book.

We can, however, diminish the labor as follows: The complete expression for v is, since $C = \frac{\delta_{\prime}}{\delta}\frac{w}{cd^2}$,

$$S(v) = \frac{\delta}{\delta_{\prime}}\frac{cd^2}{w}X + S(V);$$

in which, if metric units are employed, d will be in centimetres, w, δ and $\delta_{\prime}$ in kilogrammes, and X and V in metres.

By a proper use of the factors for converting units of the metric to those of the English system, this equation may be written

$$S(v) = 0.17215\frac{\delta d^2}{w}X + S(V);$$

in which we may take δ, d, w and X as given, in metric units; and the result will be the same as if they had been reduced to English units. In other words, the multiplier 0.17215 reduces them all at once. The coefficient of reduction C is taken at 0.9; and the standard weight of the air $\delta_{\prime}$ is taken at its mean value, 1.206 kg. per cubic metre. These being constants are included in the multiplier. The muzzle velocity, V, must be reduced to feet before taking its S-function; the resulting value of v will be in feet per second.

Example 5. Given $d = 12$ cm.; $w = 18$ kg.; $\delta = 1.220$ kg.; $V = 482.2$ m. s. $= 1582.1$ f. s.; and $X = 1450$ m., to find v. (Krupp, *Expériences de tir*, No. 60.)

$$\begin{aligned}
\text{log of multiplier} &= 9.23591\\
\log \delta &= 0.08636\\
2 \log d &= 2.15836\\
\text{a. c. } \log w &= 8.74473\\
\log X &= 3.16137\\
\hline
\log 2436.3 &= 3.38673\\
S(V) &= 4017.9\\
\hline
S(v) &= 6454.2 \qquad \therefore\ v = 1131.3 \text{ f. s.} = 344.8 \text{ m. s.}
\end{aligned}$$

Measured velocity $= 342.6$ " "

Calculated by Krupp $= 347.8$ " "

Example 6. Given $d = 30.5$ cm.; $w = 455$ kg.; $\delta = 1.274$ kg.; $V = 520.8$ m. s. $= 1708.7$ f. s.; and $X = 1900$ m., to calculate v. (Krupp, *Expériences de tir*, No. 31.)

$$
\begin{aligned}
\text{Const. log} &= 9.23591 \\
\log \delta &= 0.10517 \\
2 \log d &= 2.96860 \\
\text{a. c. } \log w &= 7.34199 \\
\log X &= 3.27875 \\
\hline
\log 852.0 &= 2.93042
\end{aligned}
$$

$$
\begin{aligned}
S(V) &= 3476.2 \\
\hline
S(v) &= 4328.2 \qquad \therefore \; v = 1513.9 \text{ f. s.} = 461.4 \text{ m. s.}
\end{aligned}
$$

Measured velocity $= 465.5$ " "
Calculated by Krupp $= 460.1$ " "

For the smooth, pointed and perfectly centred projectiles of the Hotchkiss system, the value of c is much less than for the Krupp guns, or for our new army or navy guns. For the 10-cm. rapid-fire gun, the value of c seems to be about 0.65. This value, however, needs further verification.

Striking Energy of a Projectile.—This problem is often used for determining the energy which a projectile has at any point of its trajectory; of which the following are illustrations:

Example 7. Required the striking energy of each of the projectiles of Exs. 1 and 2.

(*a*) 15-inch spherical projectile; $v = 1259$ f. s., $w = 450$ pounds.

The energy of a projectile in foot-tons is given by the equation

$$E_0 = \frac{wv^2}{4480g}.$$

For the *same* projectile the factor

$$\frac{w}{4480g}$$

is constant, and may be computed once for all. In our example the value of this factor is 0.0031233; and, therefore, the expression for the energy of a 15-in. solid shot in terms of the velocity is

$$E_0 = 0.0031233v^2.$$

The work by logarithms would be as follows:

$$\begin{aligned} \log \text{ of multiplier} &= 7.49462 \\ 2 \log v &= 6.20006 \\ \hline \log E_0 &= 3.69468 \quad \therefore E_0 = 4950.9 \text{ foot-tons.} \end{aligned}$$

(*b*) 8-inch B. L. rifle; $v = 1264.5$ f. s.; $w = 290$ pounds. For the 8-inch B. L. rifle the expression for the energy is

$$E_0 = 0.00201 28v^2;$$

whence

$$\begin{aligned} \log \text{ of multiplier} &= 7.30380 \\ 2 \log v &= 6.20384 \\ \hline \log E &= 3.50764 \quad \therefore E = 3218.4 \text{ foot-tons.} \end{aligned}$$

Example 8. Required the energy per inch of shot's circumference, in the above example.

The energy per inch of shot's circumference is generally held to be the proper measure, in comparing the relative efficiency of different guns, of the ability of a shot to penetrate armor.

Since πd is the circumference of a shot in inches, we have, designating the required energy by E_1,

$$E_1 = \frac{E_0}{\pi d} = \frac{wv^2}{4480\pi dg};$$

in which, as before, for the *same* projectile, all the factors except v^2 may be consolidated.

For the 15-inch spherical shot we have

$$E_1 = 0.000066859v^2;$$

and for the 8-inch elongated projectile weighing 290 pounds,

$$E_1 = 0.000080087v^2.$$

The answers are:

For 15-inch spherical, $E_1 = 105.98$ foot-tons;
" 8-inch elongated, $E_1 = 128.06$ " "

The following expressions for energy are applicable to all projectiles:

$$E_0 = 0.0000069407\, wv^2;$$

$$E_1 = 0.0000022093\, \frac{wv^2}{d}.$$

The logarithms of these multipliers are, respectively,

For E_0, 4.84141 — 10.
" E_1, 4.34426 — 10.

Formulæ for Striking Energy in Terms of Metric Units.—The formulæ given in the preceding article are adapted to English units only; but it frequently happens that we wish to compare two guns with respect to energy, the data of one being in English and that of the other in French units. The following formulæ give the energies in *foot-tons*, when the velocity is in metre-seconds, the weight of the projectile in kilogrammes, and the calibre in centimetres:

$$E_0 = 0.00016471\, wv^2;$$

$$E_1 = 0.00013317 \frac{wv^2}{d}.$$

The logarithms of the multipliers are 6.21672 — 10 and 6.12440 — 10, respectively.

If the velocity should be given in foot-seconds, as it would be if computed by Table I, while the weight and calibre of the projectile were given in metric units, we should have the following expressions for the energies:

$$E_0 = 0.000015302\, wv^2;$$

$$E_1 = 0.000012371 \frac{wv^2}{d}.$$

The logarithms of the multipliers are $5.18474 - 10$ and $5.09242 - 10$, respectively.

Example 9. What is the muzzle energy of a projectile fired from the Hotchkiss 10-centimetre rapid-firing gun?

For this gun we have the following data: $V = 600$ m. s.; $w = 15$ kg.; $d = 10$ cm.

$$\begin{aligned} \text{log of multiplier} &= 6.21672 \\ \log w &= 1.17609 \\ 2 \log v &= 5.55630 \\ \hline \log E_0 &= 2.94911 \quad \therefore E_0 = 889.43 \text{ foot-tons.} \end{aligned}$$

The total energy is therefore 889.43 foot-tons. To determine the energy per inch of shot's circumference we have:

$$\begin{aligned} \text{log multiplier} &= 6.12440 \\ \log wv^2 &= 6.73239 \qquad \text{(See above.)} \\ \text{a. c. } \log d &= 9.00000 \\ \hline \log E_1 &= 1.85679 \quad \therefore E_1 = 71.9 \text{ foot-tons.} \end{aligned}$$

Example 10. Compute the energy of a projectile fired from the Krupp 40-cm. gun, at a distance of 3000 metres from the gun.

For this example we have the following data: $d = 40$ cm.; $w = 920$ kg.; $V = 550$ m. s. $= 1804.5$ f. s.; $X = 3000$ m.; and $\delta = 1.206$ kg.

We must first compute the striking velocity at 3000 m. by the formula on page 21, as follows:

$$\begin{aligned} \text{const. } \log &= 9.23591 \\ \log \delta &= 0.08135 \\ 2 \log d &= 3.20412 \\ \log X &= 3.47712 \\ \text{a. c. } \log w &= 7.03621 \\ \hline \log 1083.2 &= 3.03471 \\ S(V) &= 3092.3 \\ \hline S(v) &= 4175.5 \qquad \therefore v = 1547.1 \text{ f. s.} \end{aligned}$$

For the total energy, we have

$$
\begin{aligned}
\text{const. log} &= 5.18474 \\
\log w &= 2.96379 \\
2 \log v &= 6.37904 \\
\hline
\log E_0 &= 4.52757 \qquad \therefore E_0 = 33695 \text{ foot-tons.}
\end{aligned}
$$

For the punching energy, we have

$$
\begin{aligned}
\text{const. log} &= 5.09242 \\
\log wv^2 &= 9.34283 \\
\text{a. c. } \log d &= 8.39794 \\
\hline
\log E_1 &= 2.83319 \qquad \therefore E_1 = 681 \text{ foot-tons.}
\end{aligned}
$$

Penetration of Projectiles.—To calculate the penetration of wrought-iron we will use Maitland's "Formula of 1880," which according to Mackinlay's Text-book of Gunnery, edition of 1887, is the one "now generally employed." This formula is

$$\tau = \frac{v}{608.3}\left(\frac{w}{d}\right)^{\frac{1}{2}} - 0.14d,$$

which gives the thickness (τ) of wrought-iron plate penetrated, in inches, in terms of the striking velocity, weight, and diameter of the projectile.

Example 11. How many inches of wrought-iron will the new 8-inch projectiles penetrate at 3000 yards from the gun—weight of projectile 290 pounds, muzzle velocity 1850 f. s., and $c = 0.9$?

We must first compute the striking velocity at 9000 feet from the gun, by the method already given, and then substitute this velocity in the above expression for τ. The complete work is as follows:

$$
\begin{aligned}
\log X &= 3.95424 \\
\log C &= 0.70198 \qquad \text{(Ex. 2)} \\
\hline
\log 1787.6 &= 3.25226 = \log z \\
S(V) &= 2916.9 \\
\hline
S(v) &= 4704.5 \qquad \therefore v = 1435.0 \text{ f. s.}
\end{aligned}
$$

$$
\begin{aligned}
\log w &= 2.46240 \\
\log d &= 0.90309 \\
\hline
2\,)&1.55931 \\
\hline
& 0.77965 \\
\log v &= 3.15685 \\
\text{a. c. } \log 608.3 &= 7.21588 \\
\hline
& 1.15238 = \log 14.20 \\
& 0.14d = \quad 1.12 \\
\hline
& \tau = \quad 13.08 \text{ inches.}
\end{aligned}
$$

PROBLEM II.

Given the ballistic coefficient (C) *and the remaining velocity* (v), *at a distance* (X) *from the gun, to determine the muzzle velocity* (V).

NOTE.—Hereafter C will be said to be "given" when the data upon which its value depends are supposed to be known.

Solution. Compute $S(V)$ by the equations

$$z = \frac{X}{C} \text{ and } S(V) = S(v) - z,$$

and take the value of V from the proper table.

Example 1. The velocity of a 4.5-inch solid shot fired from the M. L. siege-gun, with 8 pounds of sphero-hexagonal powder, was found to be 1387 f. s. at 101.6 feet from the gun. What was the muzzle velocity? Air normal, $c = 1$.

$$\begin{aligned} \log x &= 2.00689 \\ \log C &= 0.23764 \\ \log 58.8 &= 1.76925 = \log z \\ S(v) &= 4944.3 \\ S(V) &= 4885.5 \qquad \therefore\ V = 1399 \text{ f. s.} \end{aligned}$$

Example 2. The proposed 12-inch B. L. rifle is to fire a projectile weighing 800 pounds, which, it is expected, will penetrate 16.25 inches of solid wrought-iron armor at a distance of 3500 yards from the gun. What must be its muzzle velocity?

We have $d = 12$; $w = 800$; $X = 10500$; $c = 0.9$; and log $C = 0.79049$.

We must first determine the striking velocity necessary

to produce the required penetration. Solving Maitland's penetration formula with reference to v, we have

$$v = 608.3 \left\{ (\tau + 0.14d)\sqrt{\frac{d}{w}} \right\},$$

which becomes, by substituting for τ, d, and w their values,

$$v = 608.3 \times 17.93 \times \sqrt{\frac{12}{800}}.$$

Computation of v:

$$\begin{array}{rl}
\log 12 = & 1.07918 \\
\log 800 = & 2.90309 \\
\hline
2\,) & 8.17609 \\
\hline
 & 9.08804 \\
\log 17.93 = & 1.25358 \\
\log 608.3 = & 2.78412 \\
\hline
\log v = & 3.12574 \quad \therefore\ v = 1335.8 \text{ f. s.}
\end{array}$$

The muzzle velocity is now computed as follows:

$$\begin{array}{rl}
\log X = & 4.02119 \\
\log C = & 0.79049 \\
\hline
\log 1701.0 = & 3.23070 = \log z \\
S(v) = & 5209.0 \\
\hline
S(V) = & 3508.0 \qquad \therefore\ V = 1701 \text{ f. s.}
\end{array}$$

Example 3. Data same as in the preceding example except that the weight of the projectile is increased to 1000 pounds; and, therefore, $\log C = 0.88740$.

The striking velocity now becomes

$$v = 608 \times 17.93 \sqrt{0.012} = 1194.8 \text{ f. s.}$$

We therefore have

$$\begin{aligned} \log X &= 4.02119 \\ \log C &= 0.88740 \\ \hline \log 1360.8 = 3.13379 &= \log z \\ S(v) = 6021.9 & \\ \hline S(V) = 4661.1 & \qquad \therefore \; V = 1444 \text{ f. s.} \end{aligned}$$

It appears, then, from these last two examples that if the weight of the projectile be increased by lengthening, or otherwise, to 1000 pounds, the muzzle velocity may be diminished 257 f. s. and yet be as effective against armor at 3500 yards as with the higher velocity. It was shown by Hutton about a century ago, and has been verified by all subsequent investigators, that, the gun and charge of powder remaining the same while the weight of the projectile is made to vary, the muzzle velocities generated are very nearly inversely proportional to the square roots of the weights of the projectiles. Therefore to determine in this case what would be the muzzle velocity of the 1000 lb. projectile with the same weight of charge as gave the 800 lb. projectile a velocity of 1701 f. s., we have the proportion

$$\sqrt{1000} : \sqrt{800} :: 1701 : V;$$

$$\therefore \; V = 1701\sqrt{\frac{800}{1000}} = 1521 \text{ f. s.},$$

which is a considerably greater velocity than would be needed. The charge could therefore be reduced; but whether the strain upon the gun would be less is a question in Interior Ballistics with which we are not here concerned.

PROBLEM III.

Given the ballistic coefficient, the muzzle velocity, and the terminal or striking velocity, to determine the distance from the gun.

That is, we have C, V, and v given to compute x or X.

Solution. Take from the proper table the values of $S(V)$ and $S(v)$ for the given values of V and v. Then x is computed by the equation

$$x = C\{S(v) - S(V)\}.$$

Example 1. The muzzle velocity of a service projectile fired from the 8-inch B. L. rifle is 1850 f. s. At what distance from the gun must a target be placed in order that the striking velocity may be 1500 f. s.? (For the value of C, see Ex. 2, Prob. I.) We have

$$\begin{aligned} S(v) &= 4393.0 \\ S(V) &= \underline{2916.9} \end{aligned}$$

$$\begin{aligned} \log 1476.1 &= 3.16912 \\ \log C &= \underline{0.70198} \\ \log X &= 3.87110 \quad \therefore X = 7432.0 \text{ ft.} \end{aligned}$$

NOTE. **It will be observed that X is a particular value of x.**

Example 2. At what range will an 8-inch elongated projectile ($w = 290$ pounds) have the same energy as a 15-inch solid shot at a range of 1000 yards? The muzzle velocity of the latter is 1700 f. s., and that of the former 1850 f. s.

We have found (Ex. 7, Prob. I) that the energy of a 15-inch solid shot at 1000 yards is 4950.7 foot-tons; and that the expression for the energy of an 8-inch projectile is

$$E_0 = 0.0020128v^2.$$

As the energies are to be the same, we have

$$0.0020128v^2 = 4950.7;$$

$$\therefore\ v = \sqrt{\frac{4950.7}{0.0020128}}.$$

Employing logarithms we have

$$\begin{aligned} \log 4950.7 &= 3.69467 \\ \text{log of divisor} &= 7.30380 \\ &\ 2\,)\,\overline{6.39087} \\ \log v &= \overline{3.19543} \quad \therefore\ \boldsymbol{v = 1568.3} \text{ f. s.} \end{aligned}$$

Computation of X:

$$\begin{aligned} S(v) &= 4079.6 \\ S(V) &= 2916.9 \\ \log 1162.7 &= 3.06547 \\ \log C &= 0.70198 \\ \log X &= \overline{3.76745} \quad \therefore\ X = 5854.0 \text{ ft.} \end{aligned}$$

For all ranges, therefore, greater than 1950 yards, the striking energy of the elongated projectile *exceeds* that of the much heavier spherical shot. And this superiority of the former goes on increasing as the range becomes greater.

Example 3. The proposed 16-inch B. L. gun will fire a projectile weighing 2300 pounds with a muzzle velocity of 2000 f. s. At what distance from the gun will its energy be 55500 foot-tons? Suppose $c = 0.9$.

To determine the striking velocity we have the equation

$$\frac{wv^2}{4480g} = 55500;$$

$$\therefore\ v = \left(\frac{55500 \times 4480g}{w}\right)^{\frac{1}{2}}.$$

$$
\begin{aligned}
\log 55500 &= 4.74429 \\
\log 4480 &= 3.65128 \\
\log g &= 1.50732 \\
\text{a.c.} \log w &= 6.63827 \\
\hline
2\,)\,&6.54116 \\
\hline
\log v &= 3.27058 \qquad \therefore\ v = 1864.57 \text{ f. s.}
\end{aligned}
$$

Computation of x:

$$x = \frac{2300}{0.9 \times 256} \left\{ S(1864.57) - S(2000) \right\}$$

$$
\begin{aligned}
S(1864.57) &= 2861.7 \\
S(2000) &= 2368.2 \\
\hline
\end{aligned}
$$

$$
\begin{aligned}
\log 493.5 &= 2.69329 \\
\log 2300 &= 3.36173 \\
\text{a. c.} \log 259 &= 7.59176 \\
\text{a. c.} \log 0.9 &= 0.04576 \\
\hline
\log x &= 3.69254 \quad \therefore\ x = 4926.4 \text{ ft.}
\end{aligned}
$$

Example 4. Given $d = 20$ inches, $w = 4500$ pounds, $V = 2000$ f. s., and $E_0 = 55500$ foot-tons, to calculate X.

It will be found that the striking velocity in this case is 1333.02 f. s., and the range 11898 yards, or more than seven times as great as the range in Ex. 3.

Example 5. The Krupp 40 cm. gun, designed for the defence of Spezzia, fires a projectile 15.75 inches in diameter and weighing 2028 pounds, with a muzzle velocity of 1804.5 f. s. The new English 110-ton gun fires a projectile 16.25 inches in diameter, weighing 1800 pounds. What must be the muzzle velocity of the latter in order that its racking energy at 4000 yards may be the same as the former at the same distance?

If we make $c = 0.9$ for both guns, we shall have for the Krupp gun, $\log C = 0.95841$, and for the English gun, $\log C = 0.87932$.

First compute the striking velocity of the Krupp gun at 4000 yards = 12000 feet.

$$\begin{aligned}
\log X &= 4.07918 \\
\log C &= 0.95841 \\
\hline
\log 1320.6 &= 3.12077 \\
S(V) &= 3092.3 \\
\hline
S(v) &= 4412.9 \qquad \therefore\ v = 1495.8 \text{ f. s.}
\end{aligned}$$

As the energy of the two projectiles is to be the same, we have, in order to determine the striking velocity of the English projectile, the following equation, in which the subscripts refer to the English projectile:

$$E_0 = \frac{wv^2}{4480g} = \frac{w_1 v_1^2}{4480g};$$

$$\therefore\ v_1 = v\sqrt{\frac{w}{w_1}}.$$

Computation of v_1:

$$\begin{aligned}
\log w &= 3.30707 \\
\log w_1 &= 3.25527 \\
\hline
2)&0.05180 \\
\hline
&0.02590 \\
\log v &= 3.17487 \\
\hline
\log v_1 &= 3.20077 \qquad \therefore\ v_1 = 1587.7 \text{ f. s.}
\end{aligned}$$

Computation of V_1:

$$\begin{aligned}
\log X &= 4.07918 \\
\log C &= 0.87932 \\
\hline
\log 1584.4 &= 3.19986 = \log z \\
S(v_1) &= 3993.0 \\
\hline
S(V_1) &= 2408.6 \qquad \therefore\ V_1 = 1988.6 \text{ f. s.}
\end{aligned}$$

The English projectile will, therefore, require a muzzle velocity 184 f. s. greater than the Krupp projectile in order to have the same racking energy at 4000 yards. The energy developed at this distance is given by the above formula and is 31389 foot-tons.

Next, let us determine the muzzle velocity required by the English projectile in order that it may have the same armor-piercing energy, or energy per inch of circumference, as the Krupp projectile at 4000 yards.

In this case we have, since the energies are, by hypothesis, equal,

$$E_1 = \frac{wv^2}{4480\pi dg} = \frac{w_1 v_1^2}{4480\pi d_1 g};$$

$$\therefore\ v_1 = v\sqrt{\frac{wd_1}{w_1 d}}.$$

Computation of v_1:

$$\begin{aligned}
\log w &= 3.30707\\
\log d_1 &= 1.21085\\
\text{a. c. } \log w_1 &= 6.74473\\
\text{a. c. } \log d &= 8.80272\\
\hline
2)&\,0.06537\\
\hline
&\,0.03268\\
\log v &= 3.17487\\
\hline
\log v_1 &= 3.20755 \qquad \therefore\ v_1 = 1612.7 \text{ f. s.}
\end{aligned}$$

Computation of V_1:

$$\begin{aligned}
S(v_1) &= 3883.1\\
z &= 1584.4 \quad (z \text{ has already been computed.})\\
\hline
S(V_1) &= 2298.7 \qquad \therefore\ V_1 = 2020 \text{ f. s.}
\end{aligned}$$

PROBLEM IV.

Given V, v, and x, to determine the coefficient of reduction (c).

Solution. Take from the proper tables the values of $S(V)$ and $S(v)$. Then c is found from the equation

$$c = \frac{\delta_{\prime} w}{\delta d^2} \frac{S(v) - S(V)}{x}.$$

Example 1. At Meppen the velocity of a projectile 5.87 inches in diameter and weighing 73.855 pounds, was measured at two points of its trajectory 4662 feet apart. The velocity at the point near the gun was 1665.7 f. s., and the velocity at the farther point was 1246.74 f. s.

Observation of the atmosphere gave $\frac{\delta_{\prime}}{\delta} = 0.973$.

Required the value of c for this projectile.

$$\begin{aligned} S(v) &= 5701.3 \\ S(V) &= 3655.6 \\ \hline \log 2045.7 &= 3.31084 \\ \text{a. c. } \log x &= 6.33143 \\ \log w &= 1.86838 \\ \text{a. c. } \log d^2 &= 8.46272 \\ \log \frac{\delta_{\prime}}{\delta} &= 9.98793 \\ \hline \log c &= 9.96130 \qquad \therefore c = 0.915 \end{aligned}$$

PROBLEM V.

To determine the time from the origin (t), *when the ballistic coefficient* (C), *the horizontal distance passed over* (x), *and the muzzle velocity* (V), *are given.*

Solution. Equation (2) viz:

$$t = \frac{C}{\cos\phi}\{T(u) - T(V)\}$$

may, when ϕ is small (for reasons already given), be written

$$t = C\{T(v) - T(V)\},$$

and this, in connection with the equation (see Problem I)

$$S(v) = z + S(V),$$

solves the problem for small angles of departure.

C will be computed as already explained.

Example 1. Compute the time of flight with the data of Ex. 1, Prob. I.

Here $V = 1700$, $X = 3000$ and $\log C = 0.30869$. As the value of v has already been worked out we will not repeat the operation. We have, then, to complete our data, $v = 1259$ f. s. Therefore,

$$\begin{aligned} T(v) &= 1.445 \\ T(V) &= 0.433 \\ \hline \log 1.012 &= 0.00518 \\ \log C &= 0.30859 \\ \hline \log T &= 0.31377 \qquad \therefore \quad T = 2.06 \text{ seconds.} \end{aligned}$$

Example 2. Compute the time of flight with the data of Ex. 2, Prob. I.

We have $V = 1700$, $v = 1264.5$ and $\log C = 0.70198$.

$$\begin{aligned} T(v) &= 3.055 \\ T(V) &= 1.626 \\ \hline \log 1.429 &= 0.15503 \\ \log C &= 0.70198 \\ \hline \log T &= 0.85701 \qquad \therefore \quad T = 7.19 \text{ seconds.} \end{aligned}$$

Example 3. In firing with the 8-inch converted rifle, the observed time of flight (corrected) was 6 seconds.

If $d = 8$ inches, $w = 184$ pounds, $c = 1$ and $V = 1280$ f. s., what was the range, supposing the atmosphere to be normal?

The equations to be used are

$$T(v) = \frac{T}{C} + T(V)$$

and

$$X = C\{S(v) - S(V)\}.$$

Computation of v:

$$\begin{aligned} \log w &= 2.26482 \\ \log d^2 &= 1.80618 \\ \hline \log C &= 0.45864 \\ \log T &= 0.77815 \\ \hline \log 2.087 &= 0.31951 \\ T(V) &= 2.985 \\ \hline T(v) &= 5.072 \qquad \therefore \quad v = 992.2 \text{ f. s.} \end{aligned}$$

Computation of X:

$$
\begin{array}{r}
S(v) = 7822.8 \\
S(V) = \underline{5509.7} \\
\log 2313.1 = 3.36419 \\
\log C = \underline{0.45864} \\
\log X = 3.82283 \qquad \therefore \quad X = 6650.2 \text{ ft.}
\end{array}
$$

So far no account has been taken of the wind; that is, the air has been regarded as motionless. We will now consider the effect of a wind upon the range, time of flight and final velocity.

PROBLEM VI.

Given the ballistic coefficient (C), *the muzzle velocity* (V), *the observed time of flight* (T) *and the direction and velocity of the wind, to compute the remaining velocity* (v).

Solution. We will assume that the effect of the wind upon the velocity of a projectile, is due to that component of the wind which is parallel to the range, or plane of fire. Let β be the angle which the direction of the wind makes with the plane of fire reckoned from the target round to 180° on either side of the plane of fire. Then if W is the velocity of the wind and W_p the component parallel to the plane of fire, we shall have

$$W_p = W \cos \beta.$$

When $\beta = 0$, we have (since $\cos 0 = 1$), $W_p = W$. When $\beta = 90°$, $W_p = 0$; and when $\beta = 180°$, $W_p = -W$. For values of β between 0 and 90° the component of the wind parallel to the plane of fire retards the motion of the projectile and is positive. When β lies between 90° and 180° this component increases the projectile's motion and is negative. If $\beta = 90°$, that is if the wind blows directly across the range, it will, in accordance with our assumption, have no effect upon the velocity in the plane of fire. Having determined W_p, compute $v + W_p$ by the formula (see Appendix 1).

$$T(v + W_p) = \frac{T}{C} + T(V + W_p),$$

from which v can at once be determined. The sign of W_p depending upon the angle β, must not be overlooked in using the above equation. Should the wind come from the rear (in which case $\cos \beta$ would be negative), the above equation would become

$$T(v - W_p) = \frac{T}{C} + T(V - W_p).$$

For the use of Artillerists the velocity of the wind is required in feet per second; but as the anemometers furnished by the Government give the velocity in miles per hour, we must multiply this velocity by $\frac{44}{30}$ to reduce it to feet per second. That is

Feet per second $= \frac{44}{30}$ times miles per hour.

Example 1. The following data are taken from the record of firing with the 3.2-inch B. L. rifle (steel), at Sandy Hook, March 18, 1885: $T = 4$ seconds, $V = 1608$ f. s., $d = 3.2$ inches, $w = 13$ pounds, $W = 13.2$ f. s., $\beta = 32°$, thermometer 20°.9 and barometer 30.093 inches. As the result of a preliminary investigation we will take $c = 0.93$. Required the velocity v at the end of the 4 seconds.

We have $W_p = 13.2 \cos 32° = 11.2$ f. s., $V + W_p = 1608 + 11.2 = 1619.2$ f. s., $\frac{\delta_1}{\delta} = 0.920$, and $\log C = 0.09895$

$$\begin{aligned} \log T &= 0.60206 \\ \log C &= 0.09895 \\ \hline \log 3.185 &= 0.50311 \\ T(V + W_p) &= 1.833 \\ \hline T(v + W_p) &= 5.018 \quad \therefore \; v + W_p = 996.3;\; v = 985.1 \text{ f. s.} \end{aligned}$$

Without taking account of the wind we should have found $v = 994.0$ f. s. There is, therefore, in this case, a loss of velocity due to the wind, of about 9 f. s. in 4 seconds.

If C, V, v and W_p are given to compute T we should make use of the formula

$$T = C\{T(v + W_p) - T(V + W_p)\},$$

or

$$T = C\{T(v - W_p) - T(V - W_p)\},$$

according as the component of the wind acts against or with the projectile.

Problem VII.

Given the ballistic coefficient (C), *the muzzle velocity* (V), *the observed range* (X) *and the component of the wind parallel to the plane of fire* (W_p), *to compute the striking velocity* (v) *and time of flight* (T).

Solution. Compute the time of flight by Problem V, upon the supposition that there is no wind. The value of v will then be found by the equation (see Appendix 1).

$$S(v + W_p) = \frac{X + T W_p}{C} + S(V + W_p),$$

or

$$S(v - W_p) = \frac{X - T W_p}{C} + S(V - W_p),$$

according as the component W_p acts against or with the projectile.

The assumption made in the solution of this problem, that the time of flight is practically uninfluenced by the wind, is not strictly correct, though near enough for most practical purposes. The mean velocity of a projectile is increased or diminished by the wind in nearly the same ratio as the range; and, therefore, the time of flight is not practically affected by the wind.

Example 1. In firing with the 8-inch converted rifle the observed range was 2000 yards, with a wind blowing directly toward the gun of 30 f. s. What was the striking velocity, density of air normal?

Here $\log C = 0.45864$ (Ex. 3, Prob. V); $V = 1280$ f. s.; $W_p = 30$ f. s. and $X = 6000$ ft. To avoid confusion we will designate the value of v computed on the supposition that there is no wind by v_1.

Computation of v_1 by Problem I.

$$
\begin{aligned}
\log X &= 3.77815 \\
\log C &= 0.45864 \\
\hline
\log 2086.9 &= 3.31951 \\
S(V) &= 5509.7 \\
\hline
S(v_1) &= 7596.6 \qquad \therefore\ v_1 = 1009.7
\end{aligned}
$$

Computation of T by Problem V.

$$
\begin{aligned}
T(v_1) &= 4.846 \\
T(V) &= 2.985 \\
\hline
\log 1.861 &= 0.26975 \\
\log C &= 0.45864 \\
\hline
\log T &= 0.72839 \qquad \therefore\ T = 5.35 \text{ seconds.} \\
\log W_p &= 1.47712 \\
\hline
\log 160.5 &= 2.20551 = \log TW_p \\
X &= 6000.0 = \\
\hline
\log 6150.5 &= 3.78962 \\
\log C &= 0.45864 \\
\hline
\log 2142.8 &= 3.33098 \\
S(V+W_p) &= 5345.2 \\
\hline
S(v+W_p) &= 7488.0 \qquad \therefore\ v + W_p = 1018.7 \\
& \qquad\qquad W_p = 30.0 \\
& \qquad\qquad \therefore\ v = 988.7 \text{ f. s.}
\end{aligned}
$$

Example 2. In the above example suppose the wind to blow directly *from* the gun, the other data remaining the same. Compute the striking velocity.

Here instead of *adding* 160.5 to X as in Example 1, it must be *subtracted*.

$$
\begin{aligned}
X &= 6000.0 \\
T W_p &= \underline{160.5}
\end{aligned}
$$

$$
\begin{aligned}
\log 5839.5 &= 3.76638 \\
\log C &= \underline{0.45864} \\
\log 2031.2 &= 3.30774
\end{aligned}
$$

$$
\begin{aligned}
S(V - W_p) &= \underline{5682.1} \\
S(v - W_p) &= 7713.3
\end{aligned}
$$

$$
\begin{aligned}
\therefore v - W_p &= 1000.5 \\
W_p &= \underline{30.0} \\
\therefore v &= 1030.5
\end{aligned}
$$

Example 3. At Meppen, Nov. 26, 1880, an experimental shot was fired with the following data: $d = 10.5$ cm., $w = 16$ kg., $\delta = 1.268$ kg., $V = 467.5$ m. s. $= 1533.82$ f. s., $W = 4.2$ m. s. $= 12.73$ f. s., $\beta = 157°30'$ and $X = 1929$ m. $= 6328.9$ ft. Required the final velocity v.

To determine C in English units when the data are given, as above, in French units, we have, making $c = 0.9$ and $\delta_{,} = 1.206$ kg.,

$$C = 19.0587 \frac{w}{\delta d^2}.$$

The operation is then as follows:

$$
\begin{aligned}
\text{log of multiplier} &= 1.28009 \\
\log w &= 1.20412 \\
\text{a. c. } \log \delta &= 9.89688 \\
\text{a. c. } \log d^2 &= \underline{7.95762} \\
\log C &= 0.33871
\end{aligned}
$$

$$
\begin{aligned}
\log W &= 1.13924 \\
\log \cos \beta &= \underline{9.96562} \\
\log W_p &= 1.10486
\end{aligned}
$$

$$
\begin{aligned}
\therefore W_p &= 12.73 \text{ f. s.} \\
V - W_p &= 1521.09 \text{ f. s.}
\end{aligned}
$$

$$\log X = 3.80133$$
$$\log C = 0.33871$$
$$\log 2901.5 = 3.46262$$
$$S(V) = 4236.1$$
$$S(v_1) = 7137.6 \qquad \therefore v_1 = 1050.6 \text{ f. s.}$$
$$T(v_1) = 4.402$$
$$T(V) = 2.075$$
$$\log 2.327 = 0.36680$$
$$\log C = 0.33871$$
$$\log = 0.70551$$
$$\log W_p = 1.10486$$
$$\log 64.6 = 1.81037$$
$$X = 6328.9$$
$$\log 6264.3 = 3.79687$$
$$\log C = 0.33871$$
$$\log 2871.9 = 3.45816$$
$$S(V - W_p) = 4294.8$$
$$S(v - W_p) = 7166.7 \qquad \therefore v - W_p = 1047.7$$
$$\therefore v = 1060.4 \text{ f. s.} = 323.2 \text{ m. s.}$$
Measured velocity = 325.9 m. s.

Second Method.—If we eliminate T from the equations

$$T(v \pm W_p) = \frac{T}{C} + T(V \pm W_p)$$

and

$$S(v \pm W_p) = \frac{X \pm TW_p}{C} + S(V \pm W_p),$$

there results

$$S(v \pm W_p) \mp W_p T(v \pm W_p) = \frac{X}{C} + S(V \pm W_p) \mp W_p T(V \pm W_p),$$

from which $v \pm W_p$ can easily be found by trial, since all the terms of the second member are known quantities. By this method we find the striking velocity independently of the time of flight.

Taking the data of Ex. 1 we have, by substitution and reduction, the equation

$$S(v + 30) - 30\ T(v + 30) = 7346.4.$$

By making use of Table I, the value of $v + 30$, which satisfies this equation, is easily found to be 1018.6, the same as before. This justifies the assumption we have made, that the time of flight is not sensibly influenced by the wind; and renders this second method generally unnecessary.

PROBLEM VIII.

Given the muzzle velocity (V), *the computed range* (X), *and time of flight* (T) *to calculate the variation of the range* ($\varDelta X$) *due to a given value of* W_p.

NOTE.—Hereafter C will be supposed to be given unless otherwise stated.

Solution. Compute v by the equation

$$T(v \pm W_p) = \frac{T}{C} + T(V \pm W_p),$$

and then $\varDelta X$ by the equation

$$\varDelta X = C\{S(v + W_p) - S(V + W_p)\} - (X + TW_p),$$

or

$$\varDelta X = C\{S(v - W_p) - S(V - W_p)\} - (X - TW_p).$$

The first formula is used when the direction of the component W_p is toward the gun; in which case $\varDelta X$ is negative.

The second formula is used when the direction of W_p is toward the target; in this case $\varDelta X$ is positive.

Example 1. Compute a table of $\varDelta X$ for the 3.2-inch field-gun for a range of 1000 yards. We have $V = 1608$ f. s.; $X = 3000$ ft.; $T = 2.21$ seconds; and $\log C = 0.10843$.

We will begin by making $W_p = 10$ f. s., whence $V + W_p = 1618$ f. s.

$$\begin{aligned} \log T &= 0.34439 \\ \log C &= \underline{0.10843} \\ \log 1.722 &= 0.23596 \\ T(V + W_p) &= \underline{1.836} \\ T(v + W_p) &= 3.558 \end{aligned} \qquad \therefore\ v + W_p = 1166.5 \text{ f. s.}$$

$$Sv + W_p) = 6208.7$$
$$S(V + W_p) = \underline{3860.0}$$

$$\log 2348.7 = 3.37083$$
$$\log C = \underline{0.10843}$$

$$\log 3014.8 = 3.47926$$
$$X + TW_p = \underline{3022.1}$$

$$\Delta X = -\ 7.3 \text{ ft.}$$

In the same way may other values of ΔX be computed and arranged in a tabular form, as below. To this table are added the values of ΔX when W_p is negative, that is, when it increases the range. The remaining velocities are given in each case.

$X =$ 1000 yards $T =$ 2.21 seconds.

W_p ft. per second.	ΔX feet.	v ft. per second.	W_p ft. per second.	ΔX feet.	v ft. per second.
+ 10	− 7	1156	− 10	+ 6	1167
20	15	1151	20	13	1172
30	21	1146	30	20	1178
40	27	1140	40	25	1183
50	34	1135	50	32	1189
60	44	1130	60	38	1194

We also add a similar table for $X =$ 2000 yards and $T =$ 5.13 seconds.

$X =$ 2000 yards $T =$ 5.13 seconds.

W_p ft. per second.	ΔX feet.	v ft. per second.	W_p ft. per second.	ΔX feet.	v ft. per second.
+ 10	− 21	931	− 10	+ 35	947
20	50	922	20	63	956
30	78	914	30	90	964
40	105	906	40	119	972
50	133	897	50	147	980
60	165	889	60	174	989

It will be seen from the first of these tables that for a range of 1000 yards ΔX is approximately proportional to W_p; but this approximation decreases as the range increases, and soon ceases to be of any value as a working principle, as is shown by the second table.

Example 2. What effect would half a gale of wind (50 f. s.) blowing up or down the range, have upon the range of a four-inch projectile weighing 25 pounds and having a muzzle velocity of 1900 f. s.?

1. Suppose the wind blows up the range. Compute ΔX when $X = 1000$ yards and 2000 yards.

(*a*) $X = 3000$ ft.; $V = 1900$ f. s.; $c = 0.907$; $\frac{\delta_{\prime}}{\delta} = 1$; log $C = 0.23621$; $W_p = 50$ f. s.; $V + W_p = 1950$ f. s.

$$\begin{aligned} \log X &= 3.47712 \\ \log C &= \underline{0.23621} \\ \log 1741.4 &= 3.24091 \\ S(V) &= \underline{2729.2} \\ S(v_1) &= 4470.6 \qquad \therefore\ v_1 = 1483.565 \end{aligned}$$

$$\begin{aligned} T(v_1) &= 2.231 \\ T(V) &= \underline{1.191} \\ \log 1.040 &= 0.01703 \\ \log C &= \underline{0.23621} \\ \log T &= 0.25324 \qquad \therefore\ T = 1.79 \text{ seconds.} \end{aligned}$$

$$\begin{aligned} \frac{T}{C} &= 1.040 \\ T(V + W_p) &= \underline{1.096} \\ T(v + W_p) &= 2.136 \qquad \therefore\ v + W_p = 1513.67 \end{aligned}$$

$$S(v + W_p) = 4329.1$$
$$S(V + W_p) = 2546.4$$

$$\log 1782.7 = 3.25108$$
$$\log C = 0.23621$$

$$\log 3071.1 = 3.48729$$
$$X + TW_p = 3089.6$$

$$\Delta X = -18.5 \text{ ft.} = -6.2 \text{ yards.}$$

(*b*) $X = 6000$ ft.

$$\log X = 3.77815$$
$$\log C = 0.23621$$

$$\log 3482.9 = 3.54194$$
$$S(V) = 2729.2$$

$$S(v_1) = 6212.1 \qquad \therefore \quad v_1 = 1166.0 \text{ f. s.}$$

$$T(v_1) = 3.561$$
$$T(V) = 1.191$$

$$\log 2.370 = 0.37475$$
$$\log C = 0.23621$$

$$\log T = 0.61096 \qquad \therefore T = 4.08 \text{ seconds.}$$

$$\frac{T}{C} = 2.370$$
$$T(V + W_p) = 1.096$$

$$T(v + W_p) = 3.466 \qquad \therefore v + W_p = 1182.73$$

$$S(v + W_p) = 6100.5$$
$$S(V + W_p) = 2546.4$$

$$\log 3554.1 = 3.55073$$
$$\log C = 0.23621$$

$$\log 6122.7 = 3.78694$$
$$X + TW_p = 6204.0$$

$$\Delta X = -8.13 \text{ ft.} = -27.1 \text{ yards.}$$

Commander H. J. May, R.N., gets — 6 yards and — 28 yards, respectively, as the deviations in these two examples. (See Proceedings *Royal Artillery Institution*, Vol. XIV, page 358, Table 1.)

2. Suppose the wind blows down the range.

(*a*) $X = 3000$ ft.; $V - W_p = 1850$ f. s.

We have, as before,

$$\frac{T}{C} = 1.040$$

$$T(V - W_p) = 1.291$$

$$T(v - W_p) = 2.331 \qquad \therefore\ v - W_p = 1452.6$$

$$S(v - W_p) = 4619.0$$
$$S(V - W_p) = 2916.9$$

$$\log 1702.1 = 3.23099$$
$$\log C = 0.23621$$

$$\log 2932.3 = 3.46720$$
$$X - TW_p = 2910.4$$

$$\therefore\ \Delta X = \quad 21.9 \text{ ft.} = 7.3 \text{ yards.}$$

(*b*) $X = 6000$ ft.; $V - W_p = 1850$ f. s.

As before,

$$\frac{T}{C} = 2.370$$

$$T(V - W_p) = 1.291$$

$$T(v - W_p) = 3.661 \qquad \therefore\ v - W_p = 1149 \text{ f. s.}$$

$$S(v - W_p) = 6328.8$$
$$S(V - W_p) = 2916.9$$

$$\log 3411.9 = 3.53300$$
$$\log C = 0.23621$$

$$\log 5877.8 = 3.76921$$
$$X - TW_p = 5796.0$$

$$\therefore\ \Delta X = \quad 81.8 \text{ ft.} = 27.3 \text{ yards.}$$

Remarks upon Problems VI, VII and VIII.—The formulæ of Problems VI, VII and VIII are deduced upon the hypothesis that the effect of a wind blowing parallel to the range is simply to increase or diminish the resistance the projectile encounters. That is, if a projectile is moving nearly horizontally with a velocity v, the resistance of the air, if there is no wind, is considered proportional to v^n; but if the air has a velocity W_p parallel to the plane of fire, then the resistance is proportional to $(v + W_p)^n$, or $(v - W_p)^n$, according to the direction of W_p. (See Appendix 1.)

We have assumed in Problems VII and VIII, that the time of flight is not sensibly influenced by the wind, since the effect upon the time of the variation in the range is nearly compensated by the corresponding variation in the mean velocity.

To obtain some idea of how much the *time of flight* of a projectile propelled with a given velocity and angle of departure is effected by a change in the density of the air, we have made the following calculations with the data of the last example:

It will be shown by the next problem that the projectile of this example with a muzzle velocity of 1900 f. s. and the air at its normal density, would require an angle of departure of 2° 11′ 33″ for a range of 6000 feet, while we have already found the time of flight to be 4.08 seconds. Now (the muzzle velocity and angle of departure remaining the same), if we assume the density of the air to be two-thirds of its normal density, we must multiply C by $\frac{3}{2}$; and performing the necessary operations we shall find upon this hypothesis that $T = 4.21$ seconds, and $X = 6632$ feet. That is, $\Delta X = + 632$ ft., while ΔT is only $+ 0.13$ seconds.

Again, if we assume the air to be only one-half its normal density, we shall find $\Delta X = + 1025$ feet and $\Delta T = 0.21$ seconds.

If we suppose the air to have no density, or in other words, that the projectile moves *in vacuo*, we shall have $\Delta X = 2582$ feet, and $\Delta T = 0.44$ seconds.

On the other hand, if we assume the air to be twice its normal density, we shall find $\Delta X = -1216$ feet, and $\Delta T = -0.33$ seconds.

From these illustrations it is manifest that for flat trajectories, and values of ΔX not exceeding say 500 feet, no notice need be taken of ΔT.

Problem IX.

Given the ballistic coefficient (C), *the muzzle velocity* (V) *and angle of departure*(ϕ), *to calculate the range* (X), *the time of flight* (T), *the angle of fall* (ω) *and final velocity* (v).

SOLUTION. FIRST METHOD.

Range: From (18) we have,

$$A = \frac{\sin 2\phi}{C}.$$

With this value of A and the given muzzle velocity V, we enter Table **A** and take out the value of z corresponding to V and A. We then have

$$X = Cz.$$

We then compute u by the equation

$$S(u) = z + S(V);$$

using the value of z already found.

Time of flight: For the time of flight we have (Eq. 9)

$$T = \frac{C}{\cos\phi}\{T(u) - T(V)\};$$

which, when ϕ does not exceed 5°, may be written

$$T = C\{T(v) - T(V)\},$$

as in Problem V.

Angle of fall. We have (Eq. 19)

$$\tan\omega = \frac{BC}{2\cos^2\phi};$$

or, when ϕ does not exceed $5°$ (Eq. 20),

$$\sin 2\omega = BC.$$

B is taken from Table **B** with the arguments z (already found in determining the range) and V.

Final velocity. For the final velocity we have (Eq. 12)

$$v = u\frac{\cos\phi}{\cos\omega};$$

or, when ϕ does not exceed $5°$,

$$v = u.$$

Example 1. The 4.5-inch siege-gun, with a charge of 8 pounds of sphero-hexagonal powder, gives to a solid shot weighing 35 pounds, a muzzle velocity of 1400 f. s. With an angle of departure of $2° 50'$, what would be the range, time of flight, angle of fall and final velocity, thermometer $71°.5$ and barometer 29.59 inches?

We have $d = 4.5$, $w = 35$, $c = 1$, $\frac{\delta'}{\delta} = 1.036$, $\log C = 0.25300$, $V = 1400$, and $\phi = 2° 50'$. $\therefore\ 2\phi = 5° 40'$.

$$\begin{aligned} \log\sin 2\phi &= 8.99450 \\ \log C &= 0.25300 \\ \hline \log A &= 8.74150 \qquad \therefore A = 0.0551 \end{aligned}$$

With the arguments $V = 1400$ and $A = 0.0551$, we find from Table **A**,

$$z = 2600 + \frac{3 \times 100}{26} = 2611.5.$$

$$\begin{aligned} \log z &= 3.41689 \\ \log C &= 0.25300 \\ \hline \log X &= 3.66989 \qquad \therefore X = 4676 \text{ feet.} \end{aligned}$$

$$
\begin{aligned}
z &= 2611.5 \\
S(V) &= 4878.6 \\
\hline
S(u) &= 7490.1 \qquad \therefore u = 1018.5
\end{aligned}
$$

$$
\begin{aligned}
T(u) &= 4.741 \\
T(V) &= 2.514 \\
\hline
\log 2.227 &= 0.34772 \\
\log C &= 0.25300 \\
\hline
\log T &= 0.60072 \quad . \therefore T = 3.99 \text{ seconds.}
\end{aligned}
$$

From Table **B** we find for $V = 1400$ and $z = 2611.5$, $B = 0.0681$.

$$
\begin{aligned}
\log B &= 8.83315 \\
\log C &= 0.25300 \\
\hline
\log \sin 2\omega &= 9.08615 \quad \therefore 2\omega = 7^\circ\, 00' \quad \therefore \omega = 3^\circ\, 30'
\end{aligned}
$$

Finally, we have an account of the small values of ϕ and ω,

$$v = u = 1018 \text{ f. s.}$$

Example 2. Compute the range, etc., for the 8-inch B. L. rifle for an angle of departure of 10°.

We have $d = 8$, $w = 290$, $c = 0.9$, $V = 1850$, and $\phi = 10°$.

$$
\begin{aligned}
\log \sin 2\phi &= 9.53405 \\
\log C &= 0.70198 \\
\hline
\log A &= 8.83207 \qquad \therefore A = 0.06793
\end{aligned}
$$

Therefore, from Table **A**,

$$z = 4500 + \frac{3 \times 100}{22} = 4514$$

$$\log z = 3.65456$$
$$\log C = 0.70198$$
$$\log X = 4.35654 \qquad \therefore X = 22727 \text{ feet.}$$

$$z = 4514.0$$
$$S(V) = 2916.9$$
$$S(u) = 7430.9 \qquad \therefore u = 1023.5$$

$$T(u) = 4.685$$
$$T(V) = 1.291$$

$$\log 3.394 = 0.53071$$
$$\log C = 0.70198$$
$$\log \sec \phi = 0.00665$$
$$\log T = 1.23934 \quad \therefore T = 17.35 \text{ seconds.}$$

From Table **B** we find for $V = 1850$ and $z = 4515$, $B = 0.1010$.

$$\log B = 9.00432$$
$$\log C = 0.70198$$
$$\log 0.5 = 9.69897$$
$$2 \log \sec \phi = 0.01330$$
$$\log \tan \omega = 9.41857 \qquad \therefore \omega = 14^\circ\, 41'$$

$$\log u = 3.01009$$
$$\log \cos \phi = 9.99335$$
$$\log \sec \omega = 0.01442$$
$$\log v = 3.01786 \qquad \therefore v = 1042 \text{ f. s.}$$

Example 3. "Find the range of the proposed 20-pounde B. L. gun of 3.4-inch calibre at 7° elevation, also the angle o descent; muzzle velocity = 1650 f. s." (Proceedings of th *Royal Artillery Institution*, Vol. 15, page 364.)

This example is worked out by Niven's method in th

volume cited, by making $c = 0.9$ and jump $= 6'$. We therefore have the following data: $w = 20$, $d = 3.4$, $\phi = 7° 6'$ and $V = 1650$.

$$\begin{aligned} \log w &= 1.30103 \\ \text{a. c. } \log d^2 &= 8.93704 \\ \text{a. c. } \log c &= 0.04576 \\ \hline \log C &= 0.28383 \\ \log \sin 2\phi &= 9.38971 \\ \hline \log A &= 9.10588 \qquad \therefore A = 0.12761 \end{aligned}$$

$$\therefore\ z = 5900 + \tfrac{100}{32}(1276 - 1245) = 5996.9$$
$$\therefore B = 0.1862 + .969 \times 0.0053 = 0.1914$$

$$\begin{aligned} \log z &= 3.77793 \\ \log C &= 0.28383 \\ \hline \log X &= 4.06176 \quad \therefore\ X = 11528 \text{ feet.} \end{aligned}$$

By Niven's Method 11515 feet.

Difference $= 13$ feet.

$$\begin{aligned} \log B &= 9.28194 \\ \log C &= 0.28383 \\ \log 0.5 &= 9.69897 \\ 2 \log \sec. \phi &= 0.00669 \\ \hline \log \tan \omega &= 9.27143 \quad \therefore\ \omega = 10°35' \end{aligned}$$

By Niven's Method 10 37

Difference $= 2'$

Example 4. Compute the range for the Krupp 24-cm. gun, with the following data : $d = 24$ cm., $w = 215$ kg., $V = 529$ m. s. $= 1735.6$ f. s., $\phi = 8° 35'$ and $\delta = 1.262$ kg.

$$\begin{aligned}
\text{Const log} &= 1.28009 \quad \text{(See Ex. 3, Prob. VII.)}\\
\log w &= 2.33244\\
\text{a. c. } \log d^2 &= 7.23958\\
\text{a. c. } \log \delta &= 9.89894\\
\hline
\log C &= 0.75105\\
\log \sin 2\phi &= 9.47005\\
\hline
\log A &= 8.71900 \quad \therefore\ A = 0.0524
\end{aligned}$$

$$\therefore\ z = 3400 + \frac{100}{21}\left\{\frac{35.6}{50} \times 30 + 524 - 536\right\} = 3444$$

$$\begin{aligned}
\log z &= 3.53706\\
\log C &= 0.75105\\
\hline
\log X &= 4.28811 \quad \therefore\ X = 19414 \text{ feet.}
\end{aligned}$$

Measured range = 19567 feet.

Difference = 153 feet.

(*Expériences de tir*, No. 56.)

Example 5. Compute the range for the Krupp 40-cm. gun, with the following data: $d = 40$ cm., $w = 920$ kg., $V = 550$ m. s. $= 1804.5$ f. s., quadrant elevation $5° 21'$, jump $= 14'$, $\phi = 5° 21' + 14' = 5° 35'$ and $\delta = 1.206$.

$$\begin{aligned}
\text{Const. log} &= 1.28009\\
\log w &= 2.96379\\
\text{a. c. } \log d^2 &= 6.79588\\
\text{a. c. } \log \delta &= 9.91865\\
\hline
\log C &= 0.95841\\
\log \sin 2\phi &= 9.28705\\
\hline
\log A &= 8.32864 \quad \therefore\ A = 0.0213
\end{aligned}$$

$$\therefore z = 1800 + \frac{100}{14}\left\{\frac{4.5}{50} \times 10 + 213 - 213\right\} = 1806$$

$$\begin{aligned} \log z &= 3.25672 \\ \log C &= 0.95841 \\ \hline \log X &= 4.21513 \quad \therefore X = 16411 \text{ feet.} \end{aligned}$$

$$\begin{aligned} \text{Measured range} &= 16391 \text{ feet.} \\ \hline \text{Difference} &= 20 \text{ feet.} \end{aligned}$$

Example 6. Compute the range with the data of Ex. 5, except that $\phi = 15°$.

$$\begin{aligned} \log \sin 2\phi &= 9.69897 \\ \log C &= 0.95841 \\ \hline \log A &= 8.74056 \quad \therefore A = 0.0550 \end{aligned}$$

$$\therefore z = 3700 + \frac{100}{21}\left\{\frac{4.5}{50} \times 29 + 550 - 537\right\} = 3784$$

$$\begin{aligned} \log z &= 3.57795 \\ \log C &= 0.95841 \\ \hline \log X &= 4.53636 \quad \therefore X = 34384 \text{ ft.} = 6.512 \text{ miles.} \end{aligned}$$

Example 7. Compute the range of the proposed 16-inch B. L. rifle with the following data: $d = 16$ inches, $w = 2300$ pounds, $c = 0.9$, $V = 1850$ f. s. and $\phi = 15°$.

$$\begin{aligned} \log \sin 2\phi &= 9.69897 \\ \log C &= 0.99925 \\ \hline \log A &= 8.69972 \quad \therefore A = 0.0501 \end{aligned}$$

$$\therefore z = 3600 + \frac{100}{19}(501 - 489) = 3663$$

$$\begin{aligned}\log z &= 3.56384\\ \log C &= 0.99925\\ \hline \log X &= 4.56309 \quad \therefore X = 36567 \text{ ft.} = 6.926 \text{ miles.}\end{aligned}$$

In the last two examples the projectiles would attain altitudes of more than 3000 feet, and consequently the actual ranges would be somewhat greater than those computed, on account of the less resistance the projectile meets with at high altitudes. This will be considered in a subsequent problem.

SECOND METHOD.

We have not considered it worth while to compute auxiliary tables for spherical projectiles, on account of their less frequent use; and, therefore, for this class of projectiles, in the absence of tables, we proceed as follows:

We have from (10) the following relation, in which, of course, u refers to the point of fall, where $y = 0$:

$$\frac{A(u) - A(V)}{S(u) - S(V)} = \frac{\sin 2\phi}{C} + I(V)$$

The second member of this equation consists entirely of known quantities; and in the first member $A(V)$ and $S(V)$ are known. But as the relation between the S-functions and A-functions does not admit of a direct solution of this equation it is necessary to determine u by trial. We may deduce a near value of u, and one sufficiently accurate in many cases of curved fire, by the following method:

We have from the origin to the summit, by (2),

$$t_0 = \frac{C}{\cos\phi}\{T(u_0) - T(V)\};$$

and from the origin to the point of fall,

$$T = \frac{C}{\cos\phi}\{T(u) - T(V)\}.$$

If we assume (what is approximately true) that

$$T = 2t_0,$$

we shall have, from the above equations,

$$T(u) = 2T(u_0) - T(V);$$

which gives u by means of the T-functions, u_0 being computed by Equation (6), viz.,

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V).$$

This value of u is always too great and makes the first member of the equation

$$\frac{A(u) - A(V)}{S(u) - S(V)} = \frac{\sin 2\phi}{C} + I(V, = I(u_0)$$

a little too small; but the exact value (that is, the value that satisfies the above equation) can readily be found by the rule of double position, as will be illustrated in the following examples:

Example 8. Calculate u with the data of Ex. 2, without using the auxiliary tables.

First compute u_0.

$$\begin{aligned} \sin 2\phi &= 9.53405 \\ \log C &= 0.70198 \\ \hline \log 0.06793 &= 8.83207 \\ I(V) &= 0.03727 \\ \hline I(u_0) &= 0.10520 \qquad \therefore\ u_0 = 1299.2 \end{aligned}$$

Next taking the values of $S(V)$ and $A(V)$ from Table I, we have the equation

$$\frac{A(u) - 46.93}{S(u) - 2916.9} = 0.10520,$$

from which to determine u. It frequently happens that an approximate value of u is known beforehand. But in the absence of such knowledge it is best to calculate an approximate value by means of the T-functions. We have

$$\begin{aligned} T(u_0) &= 2.903 \\ 2T(u_0) &= 5.806 \\ T(V) &= 1.291 \\ \hline T(u) &= 4.515 \qquad \therefore\ u = 1039 \end{aligned}$$

Introducing this value of u into the first member of the above equation, it becomes

$$\frac{487.04 - 46.93}{7258.6 - 2916.9} = \frac{440.11}{4341.7} = 0.10137,$$

which is too small, and the error is

$$0.10520 - 0.10137 = +0.00383.$$

We will now diminish the first assumed value of u by 20 and take $u = 1019$ for a second trial. This gives

$$\frac{532.66 - 46.93}{7483.9 - 2916.9} = \frac{485.73}{4567.0} = 0.10636,$$

which is too large, and the new error is

$$0.10520 - 0.10636 = -0.00116.$$

We now get the correct value of u by means of the following proportion: As the difference (algebraic) of the errors is to the difference of the assumed values of u, so is the lesser of the two errors (numerically) to the correction to be applied to the corresponding assumed value of u. As one of the errors is positive and the other negative, their algebraic difference is their numerical sum. Therefore we have the proportion

$$499 : 20 :: 116 : 4.6$$

$$\therefore\ u = 1019 + 4.6 = 1023.6,$$

which agrees within one-tenth of a foot with the value of u computed by means of the auxiliary tables.

The range and time of flight are computed by methods already given. The angle of fall is computed by (16), viz.,

$$\tan \omega = \frac{C}{2 \cos^2 \phi} \{I(u) - I(u_0)\};$$

$$\begin{aligned} I(u) &= 0.20612 \\ I(u_0) &= 0.10520 \\ \log 0.10092 &= 9.00398 \\ \log \frac{C}{2 \cos^2 \phi} &= 0.41425 \\ \log \tan \omega &= 9.41823 \quad \therefore\ \omega = 14^\circ\ 41' \end{aligned}$$

Example 9. What will be the range, etc., of a solid shot fired from the 15-inch S. B. gun, with a muzzle velocity of 1700 f. s. and angle of departure of 8°, air normal?

The operation is as follows:

$$\begin{aligned} \log \sin 2\phi &= 9.44034 \\ \log C &= 0.30859 \\ \log 0.13544 &= 9.13175 \\ I(V) &= 0.01517 \\ I(u_0) &= 0.15061 \qquad \therefore\ u_0 = 955.4 \text{ f. s.} \end{aligned}$$

$$\begin{aligned} T(u_0) &= 2.956 \\ 2T(u_0) &= 5.912 \\ T(V) &= 0.433 \\ \hline T(u) &= 5.479 \qquad \therefore\ u = 747 \text{ f. s.} \end{aligned}$$

As the value of u determined by the T-functions is too great by from 10 to 30 f. s., we will assume for a first trial $u = 735$. Therefore, taking out the proper numbers from Table II, we have

$$\frac{788.91 - 5.73}{6187 - 798} = \frac{783.18}{5389} = 0.14533.$$

Therefore first trial error $= 0.15061 - 0.14533 = +0.00528$.

Next assume $u = 715$, and we have

$$\frac{897.96 - 5.73}{6472 - 798} = \frac{892.23}{5674} = 0.15725.$$

Second trial error $= 0.15061 - 0.15725 = -0.00624$,

$$\therefore\ 1192 : 20 :: 528 : 8.9$$

$$\therefore\ u = 735 - 8.9 = 726.1 \text{ f. s.}$$

$$\begin{aligned} S(u) &= 6311 \\ S(V) &= 798 \\ \hline \log 5513 &= 3.74139 \\ \log C &= 0.30859 \\ \hline \log X &= 4.04998 \qquad \therefore\ X = 11220 \text{ feet.} \end{aligned}$$

$$\begin{aligned} T(u) &= 5.873 \\ T(V) &= 0.433 \\ \hline \log 5.440 &= 0.73560 \\ \log C &= 0.30859 \\ \log \sec \phi &= 0.00425 \\ \hline \log T &= 1.04844 \qquad \therefore\ T = 11.18 \text{ seconds.} \end{aligned}$$

$$
\begin{aligned}
I(u) &= 0.37957 \\
I(u_0) &= 0.15061 \\
\hline
\log 0.22896 &= 9.35976 \\
\log C &= 0.30849 \\
\log 0.5 &= 9.69897 \\
2 \log \sec \phi &= 0.00850 \\
\hline
\log \tan \omega &= 9.37582 \qquad \therefore \omega = 13^\circ\, 22 \\
\\
\log u &= 2.86100 \\
\log \cos \phi &= 9.99575 \\
\log \sec \omega &= 0.01193 \\
\hline
\log v &= 2.86868 \qquad \therefore v = 739 \text{ f. s.}
\end{aligned}
$$

PROBLEM X.

Given the range X and angle of departure (ϕ), to compute the muzzle velocity (V).

SOLUTION. (FIRST METHOD.)

Compute A and z by the formulæ

$$A = \frac{\sin 2\phi}{C}$$

and

$$z = \frac{X}{C},$$

and then take from Table **A** the corresponding value of V, which is the velocity required.

Example 1. With a quadrant elevation of $7°27'$ the range with the 40-cm. Krupp gun was 6588 metres, or 21614.6 ft. If the jump was $14'$ and air normal, what was the muzzle velocity?

We have $\log C = 0.95841$ and $\phi = 7° \, 41'$

$$\begin{aligned} \log \sin 2\phi &= 9.42324 \\ \log C &= 0.95841 \\ \hline \log A &= 8.46483 \qquad \therefore A = 0.0292 \end{aligned}$$

$$\begin{aligned} \log X &= 4.33475 \\ \log C &= 0.95841 \\ \hline \log z &= 3.37634 \qquad \therefore z = 2378.7 \end{aligned}$$

Therefore from Table **A**

$$V = 1800 + \frac{50}{15}\left\{\frac{78.7}{100} \times 16 + 287 - 292\right\} = 1825 \text{ f. s.}$$

The measured velocities ranged from 1798 f. s. to 1828 f. s.

Expériences de tir, No. 63.

Example 2. The average range of five shots fired at Sandy Hook, Sept. 29, 1885, from the 12-inch experimental cast-iron B. L. rifle, with a quadrant elevation of 4°, was 11089.8 feet, weight of projectile 80 opounds, thermometer 72°.4, barometer 30.115 inches. What was the muzzle velocity?

We have $d = 12$, $w = 800$, $c = 0.9$, $\frac{\delta_{\prime}}{\delta} = 1.020$, $\log C = 0.79909$, $X = 11089.8$ and $\phi = 4° + \text{jump} = 4° \, 16'$, say.

$$\begin{aligned} \log \sin 2\phi &= 9.17139 \\ \log C &= 0.79909 \\ \hline \log A &= 8.37230 \qquad \therefore A = 0.0236 \end{aligned}$$

$$\begin{aligned} \log X &= 4.04492 \\ \log C &= 0.79909 \\ \hline \log z &= 3.24583 \qquad \therefore z = 1761.3 \end{aligned}$$

$$\therefore V = 1650 + \frac{50}{14}\left\{\frac{61.3}{100} \times 16 + 238 - 236\right\} = 1692 \text{ f. s.}$$

which agrees almost exactly with the mean of the measured velocities, reduced to muzzle velocity. See Report *Chief of Ordnance* for 1885, page 140.

Example 3. At Sandy Hook, March 18, 1885, ten shots were fired from the 3.2-inch B. L. rifle (steel) to determine the range for 4° quadrant elevation with the following data: $X = 7092.6$ ft., thermometer 25°, barometer 29.974 inches, jump 22′, $w = 13$ pounds and $d = 3.2$ inches. We therefore have $\phi = 4° \, 22'$ $\frac{\delta_{\prime}}{\delta} = 0.932$ and $\log C = 0.10458$. What was the muzzle velocity?

Note.—For this gun $c = 0.93$.

$$\begin{aligned} \log \sin 2\phi &= 9.18137 \\ \log C &= 0.10458 \\ \hline \log A &= 9.07679 \qquad \therefore A = 0.1193 \end{aligned}$$

$$\begin{aligned}\log X &= 3.85081\\ \log C &= 0.10458\\ \hline \log z &= 3.74623 \qquad \therefore\ z = 5574.8\end{aligned}$$

$$\therefore\ V = 1600 + \frac{50}{59}\left\{\frac{74.8}{100} \times 32 + 1181 - 1193\right\} = 1611 \text{ f. s.}$$

Example 4. Data same as above, except that $\phi = 6° +$ jump $= 6° \ 23'$, $X = 9108.9$ ft., thermometer 26° and barometer 29.726 inches.

It will be found that $\frac{\delta_{\prime}}{\delta} = 0.942$, $\log C = 0.10921$, $A = 0.17185$ and $z = 7083.6$

$$\therefore\ V = 1600 + \frac{50}{76}\left\{\frac{83.6}{100} \times 37 + 1690 - 1718.5\right\} = 1602 \text{ f. s.}$$

These velocities agree very closely with those determined by experiment.

SECOND METHOD.

The muzzle velocity may also be determined when the range and angle of departure are known, without the help of the auxiliary tables, as follows:

We have the equations

$$S(u) = z + S(V)$$

and

$$\sin 2\phi = C\left\{\frac{A(u) - A(V)}{z} - I(V)\right\}$$

from which to determine, by trial, a value of V which will satisfy these equations,—C, z, and ϕ being given. The best way of accomplishing this will be shown by examples.

Example 5. Wishing to ascertain the muzzle velocity of a shell fired from the 10-inch S. B. gun with a charge of twenty pounds of cannon powder; and it being impracticable, from the position of the gun, to use a chronograph, the following data were selected from the record of target practice at Fort

Monroe, July 7, 1887, for the purpose of determining the velocity by calculation: Range (mean of four shots), 6795 feet; quadrant elevation 6°; weight of shell 107 pounds; thermometer 82°, and barometer 29.950 inches.

We have $X = 6795$; $\phi = 6° + \text{jump} = 6°\ 10'$, say; $d = 9.87$; $w = 107$; $\frac{\delta_{\prime}}{\delta} = 1.045$; $c = 1$, and $\log C = 0.05986$.

The operation is as follows:

$$\begin{aligned} \log X &= 3.83219 \\ \log C &= 0.05986 \\ \hline \log z &= 3.77233 \qquad \therefore z = 5920 \end{aligned}$$

For a first trial assume $V = 1500$.

$$\begin{aligned} S(V) &= 1413 \\ z &= 5920 \\ \hline S(u) &= 7333 \qquad \therefore u = 661.2 \end{aligned}$$

$$\begin{aligned} A(u) &= 1291.1 \\ A(V) &= 19.6 \\ \hline \log 1271.5 &= 3.10431 \\ \log z &= 3.77233 \\ \hline \log 0.21477 &= 9.33198 \\ I(V) &= 0.03072 \\ \hline \log 0.18405 &= 9.26494 \\ \log C &= 0.05986 \\ \hline \log \sin 2\phi &= 9.32480 \end{aligned}$$

But $\phi = 6°\ 10'$, $2\phi = 12°\ 20'$, and logs in $2\phi = 9.32960$. Therefore the error due to the first trial is

$$9.32960 - 9.32480 = +0.00480.$$

As the angle of departure, upon the assumption that $V = 1500$ f. s., is too small, it follows that we have taken V too great. We will therefore assume for a second trial $V = 1480$.

$$\begin{aligned} S(V) &= 1479 \\ z &= 5920 \\ \hline S(u) &= 7399 \qquad \therefore u = 657.44 \end{aligned}$$

$$\begin{aligned} A(u) &= 1325.5 \\ A(V) &= 21.7 \\ \hline \log 1303.8 &= 3.11521 \\ \log z &= 3.77233 \\ \hline \log 0.22023 &= 9.34288 \\ I(V) &= 0.03262 \\ \hline \log 0.18761 &= 9.27326 \\ \log C &= 0.05986 \\ \hline \log \sin 2\phi &= 9.33312 \end{aligned}$$

$$\therefore \text{error} = 9.32960 - 9.33312 = -0.00352$$

Therefore we have the proportion (see page (64)

$$877 : 20 :: 352 : 8.$$

$$\therefore V = 1480 + 8 = 1488 \text{ f. s.,}$$

and this value of V satisfies both the above equations.

Example 6. Compute the muzzle velocity of a 10-inch solid shot fired with 20 pounds of cannon powder, with the following data, taken from the Fort Monroe records of 1887: $X = 6379$ feet, $\phi = 5^\circ\, 45'$, $d = 9.87$ inches, $w = 128$ pounds, $\frac{\delta_1}{\delta} = 1.050$ and $\log C = 0.13976$.

$$\begin{aligned} \log X &= 3.80475 \\ \log C &= 0.13976 \\ \hline \log z &= 3.66499 \qquad \therefore z = 4624 \end{aligned}$$

As the charge is the same as in Ex. 5, we may determine an approximate value of V by Hutton's law. We have by this law

$$V = 1488\sqrt{\frac{107}{128}} = 1360.4 \text{ f. s.}$$

We will therefore assume $V = 1365$ for a first trial.

$$\begin{aligned} S(V) &= 1876 \\ z &= 4624 \\ \hline S(u) &= 6500 \qquad \therefore u = 713.1 \end{aligned}$$

$$\begin{aligned} A(u) &= 909.20 \\ A(V) &= 37.21 \\ \hline \log 871.99 &= 2.94051 \\ \log z &= 3.66499 \\ \hline \log 0.18859 &= 9.27552 \\ I(V) &= 0.04530 \\ \hline \log 0.14329 &= 9.15622 \\ \log C &= 0.13976 \\ \hline \log \sin 2\phi &= 0.29598 \end{aligned}$$

But $\log \sin 2\phi = \log \sin 11^\circ\ 30' = 9.29966.$

$$\therefore \text{error} = +0.00368.$$

Next try $V = 1355$:

$$\begin{aligned} S(V) &= 1913 \\ z &= 4624 \\ \hline S(u) &= 6537 \qquad \therefore u = 710.6 \end{aligned}$$

$$
\begin{array}{rl}
A(u) = & 924.26 \\
A(V) = & \underline{38.88}
\end{array}
$$

$$
\begin{aligned}
\log 885.38 &= 2.94713 \\
\log z &= \underline{3.66499}
\end{aligned}
$$

$$
\log 0.19149 = 9.28214
$$

$$
I(V) = 0.04656
$$

$$
\begin{aligned}
\log 0.14493 &= 9.16116 \\
\log C &= \underline{0.13976}
\end{aligned}
$$

$$
\log \sin 2\phi = 9.30092
$$

$$
\therefore \text{error} = -0.00126;
$$

$$
\therefore 494 : 10 :: 126 : 2.5.
$$

$$
\therefore V = 1355 + 2.5 = 1357.5 \text{ f.s.},
$$

differing but three feet from that deduced by Hutton's law.

PROBLEM XI.

Given the range (X) *and final velocity* (v), *to compute the muzzle velocity* (V), *the angle of departure* (ϕ), *the angle of fall* (ω), *and time of flight* (T).

Solution for small angles of departure. Compute V, as in Problem II, by the equation

$$S(V) = S(v) - z.$$

Then with V and z as arguments take out A and B from the auxiliary tables. We then have

$$\sin 2\phi = AC$$

and

$$\sin 2\omega = BC.$$

The time of flight can be computed (as in Problem V) by the equation

$$T = C\{T(v) - T(V)\}.$$

Example 1. Compute ϕ ω, and T with the data of Ex. 2, Prob. II.

We have given, $\log C = 0.79049$, $X = 10500$, $v = 1335.8$, $V = 1701$, and $z = 1701$; and from the tables we get $A = 0.0224$, $B = 0.0263$, $T(V) = 1.624$ and $T(v) = 2.755$. We then proceed as follows:

$$\begin{aligned} \log A &= 8.35025 \\ \log C &= 0.79049 \\ \hline \log \sin 2\phi &= 9.14074 \qquad \therefore \phi = 3^\circ\ 59' \\ \\ \log B &= 8.41996 \\ \log C &= 0.79049 \\ \hline \log \sin 2\omega &= 9.21045 \qquad \therefore \omega = 4^\circ\ 40' \end{aligned}$$

$$T(v) = 2.755$$
$$T(V) = 1.624$$
$$\log 1.131 = 0.05346$$
$$\log C = 0.79049$$
$$\log T = 0.84395 \quad \therefore \; T = 6.98 \text{ seconds.}$$

Example 2. Compute ϕ, ω, and T with the data of Ex. 3, Prob. II.

We have given, $X = 10500$, $\log C = 0.88740$, $V = 1444$, $v = 1194.8$, and $z = 1360.8$; and from the tables we find $A = 0.0240$, $B = 0.0274$, $T(V) = 2.360$, and $T(v) = 3.400$.

By calculations precisely like those of Ex. 1, we find

$$\phi = 5^\circ\, 20',$$
$$\omega = 6^\circ\, 6',$$
$$T = 8.02 \text{ seconds.}$$

In these two examples ϕ and ω are small angles, and the ratio of their cosines nearly unity; so that v can be taken for u without perceptible error. In the last example, for instance, the difference between v and u at the point of fall is less than 2 feet; and this would not appreciably affect the value of ϕ. But for the higher angles of direct fire the error involved in taking v for u would be too great to be neglected, as is shown by the following example:

Example 3. Given, $\log C = 0.70198$, $X = 22727$ feet, and $v = 1042$ f. s., to compute ϕ. (See Ex. 2, Prob. IX.)

First compute the muzzle velocity:

$$\log X = 4.35654$$
$$\log C = 0.70198$$
$$\log z = 3.65456$$
$$\therefore z = 4514.0$$
$$S(v) \doteq 7226.6$$
$$S(V) = 2712.6 \qquad \therefore V = 1904.5$$

$$\therefore A = 0.0610 + \frac{14}{100} \times .0021 - \frac{4.5}{50} \times .0030 = 0.0610$$

$$\therefore B = 0.0960 + \frac{14}{100} \times .0038 - \frac{4.5}{50} \times .0043 = 0.0961$$

$$\begin{aligned} \log A &= 8.78533 \\ \log C &= 0.70198 \\ \hline \log \sin 2\phi &= 9.48731 \qquad \therefore \phi = 8^\circ\ 57' \end{aligned}$$

$$\begin{aligned} \log B &= 8.98272 \\ \log C &= 0.70198 \\ 2 \log \sec \phi &= 0.01064 \\ \log 0.5 &= 9.69897 \\ \hline \log \tan \omega &= 9.39431 \qquad \therefore \omega = 13^\circ\ 55' \end{aligned}$$

The true values of V, ϕ, and ω are, respectively, 1850, 10°, and 14° 41′; so that the above values, deduced upon the supposition that $u = v$, can hardly be considered as approximations even. But, though the angles ϕ and ω, as computed above, differ considerably from their true values, the *ratio of their cosines* is very nearly what it should be; and this enables us to determine a practically correct value of the auxiliary u by the equation

$$u = v \frac{\cos \omega}{\cos \phi}$$

as follows:

$$\begin{aligned} \log v &= 3.01787 \\ \log \cos \omega &= 9.98706 \\ \log \sec \phi &= 0.00532 \\ \hline \log u &= 3.01025 \qquad \therefore u = 1024 \text{ f. s.} \end{aligned}$$

With this value of u the computed value of v becomes very nearly what it should be, as we see below.

$$\begin{aligned} S(u) &= 7425.5 \\ z &= 4514.0 \\ \hline S(V) &= 2911.5 \qquad \therefore\ V = 1851.4 \text{ f. s.} \end{aligned}$$

$$\therefore\ A = 0.0676 + .14 \times .0022 - .028 \times .0034 = 0.0678.$$

$$\therefore\ B = 0.1005 + .14 \times .0039 - .028 \times .0045 = 0.1009,$$

and these values of A and B are practically correct.

PROBLEM XII.

Given the muzzle velocity (V) *and range* (X), *to compute the trajectory.*

SOLUTION. (FIRST METHOD.)

Compute z by the equation

$$z = \frac{X}{C},$$

and then with the arguments V and z take the corresponding values of A and B from the auxiliary tables. We then have for the angle of departure, by (18),

$$\sin 2\phi = AC.$$

The angle of fall can be computed by (19) or (20), according to the value of ϕ.

The following is, however, preferable in all cases: If we divide (18) by (19) and reduce, we shall get

$$\tan \omega = \frac{B}{A} \tan \phi.$$

For the final velocity we have

$$S(u) = z + S(V)$$

and

$$v = u\frac{\cos \phi}{\cos \omega}.$$

The time of flight is computed by the equation

$$T = \frac{C}{\cos \phi}\{T(u) - T(V)\},$$

with the usual modifications for small values of ϕ.

Example 1. The 100-ton gun, tried at Spezzia in 1880, fired a projectile 44.6 cm. in diameter and weighing 1000 kg. with a muzzle velocity of 450 m. s. = 1476.4 f. s.

Calculate the angle of departure for a range of 4133 m. = 13560 feet, supposing the air to be normal.

We have $d = 44.6$, $w = 1000$, $\delta = 1.206$, $V = 1476.4$, and $X = 13560$.

Computation of z:

$$\begin{aligned} \text{Const. log} &= 1.28009 \\ \log w &= 3.00000 \\ \text{a. c. } \log d^2 &= 6.70134 \\ \text{a. c. } \log \delta &= 9.91865 \\ \hline \log C &= 0.90008 \\ \log X &= 4.13226 \\ \hline \log z &= 3.23218 \qquad \therefore\ z = 1706.8 \end{aligned}$$

Angle of departure. We have from Table **A**,

$$A = 0.0307 + .068 \times .0021 - .528 \times .0021 = 0.0297.$$

$$\begin{aligned} \log A &= 8.47276 \\ \log C &= 0.90008 \\ \hline \log \sin 2\phi &= 9.37284 \qquad \therefore\ \phi = 6^\circ\ 49' \end{aligned}$$

The angle actually employed was 6° 45′. The difference (4′) may fairly be attributed to the jump of the piece, or to atmospheric conditions not given.

Angle of fall. We have from Table **B**,

$$B = 0.0360 + .068 \times .0028 - .528 \times .0022 = 0.0350.$$

$$\begin{aligned} \log \tan \phi &= 9.07751 \\ \log B &= 8.54407 \\ \text{a. c. } \log A &= 1.52724 \\ \hline \log \tan \omega &= 9.14882 \qquad \therefore\ \omega = 8^\circ\ 1' \end{aligned}$$

Example 2. According to the range-table, the 8-inch converted rifle requires an angle of departure of $5° 43'$ and muzzle velocity of 1404 f. s., to attain a range of 3000 yards. But owing to deterioration of the powder, suppose the actual range to be only 2700 yards. How much should the angle of departure be increased to bring the range up to 3000 yards?

Here $d = 8$, $w = 183$, $\phi = 5° 43'$, $c = 1$, and $\log C = 0.45627$.

We have, first, to compute the actual muzzle velocity from the observed range and angle of departure, by means of Problem X., as follows:

$$\begin{aligned} \log \sin 2\phi &= 9.29716 \\ \log C &= 0.45627 \\ \hline \log A &= 8.84089 \qquad \therefore A = 0.06933 \end{aligned}$$

$$\begin{aligned} \log X &= 3.90849 \\ \log C &= 0.45627 \\ \hline \log z &= 3.45222 \qquad \therefore z = 2832.9 \end{aligned}$$

$$\therefore V = 1250 + \frac{50}{46}\{.329 \times 32 + 728 - 693\} = 1299.5.$$

We now have $V = 1299.5$ and $X = 9000$ to find ϕ.

$$\begin{aligned} \log X &= 3.95424 \\ \log C &= 0.45627 \\ \hline \log z &= 3.49797 \qquad \therefore z = 3147.5 \end{aligned}$$

$$\therefore A = 0.0823 + .475 \times .0032 - .99 \times .0051 = 0.0788.$$

$$\begin{aligned} \log A &= 8.89653 \\ \log C &= 0.45627 \\ \hline \log \sin 2\phi &= 9.35280 \qquad \therefore \phi = 6° 31' \end{aligned}$$

The angle of departure must therefore be increased $6° 31' - 5° 43' = 43'$.

SECOND METHOD.

(Without the use of the auxiliary tables.)

Compute u and u_0 by the equations

$$z = \frac{X}{C},$$

$$S(u) = z + S(V),$$

and

$$I(u_0) = \frac{A(u) - A(V)}{z}.$$

Then we have

$$\sin 2\phi = C\{I(u_0) - I(V)\},$$

$$\tan \omega = \frac{C}{2\cos^2\phi}\{I(u) - I(u_0)\},$$

$$T = \frac{C}{\cos\phi}\{T(u) - T(V)\}.$$

Example 3. With a charge of 20 pounds of cannon powder the 10-inch S. B. gun ($d = 9.87$ inches) fires shot and shell as follows: shot weighing 128 pounds, with a muzzle velocity of 1358 f. s.; shell weighing 107 pounds, with a muzzle velocity of 1493 f. s. Compare the trajectories of the two kinds of projectiles for a range of 2000 yards.

For the shot we have $d = 9.87$, $w = 128$, $\log C = 0.11858$, $X = 6000$ and $V = 1358$.

Computation of u:

$$\begin{aligned} \log X &= 3.77815 \\ \log C &= 0.11858 \\ \hline \log z &= 3.65957 \\ \therefore\ z &= 4566.4 \\ S(V) &= 1901.4 \\ \hline S(u) &= 6467.8 \qquad \therefore\ u = 715.3 \text{ f. s.} \end{aligned}$$

Computation of ϕ:

$$
\begin{aligned}
A(u) &= 896.23 \\
A(V) &= 38.37 \\
\hline
\log 857.86 &= 2.93342 \\
\log z &= 3.65957 \\
\hline
\log I(u_0) &= 9.27385 \\
\therefore\ I(u_0) &= 0.18787 \\
I(V) &= 0.04617 \\
\hline
\log 0.14170 &= 9.15137 \\
\log C &= 0.11858 \\
\hline
\log \sin 2\phi &= 9.26995 \qquad \therefore\ \phi = 5^\circ\ 22'
\end{aligned}
$$

Computation of ω:

$$
\begin{aligned}
I(u) &= 0.39890 \\
I(u_0) &= 0.18787 \\
\hline
\log 0.21103 &= 9.32434 \\
\log C &= 0.11858 \\
2 \log \sec \phi &= 0.00382 \\
\log 0.5 &= 9.69897 \\
\hline
\log \tan \omega &= 9.14571 \qquad \therefore\ \omega = 7^\circ\ 58'
\end{aligned}
$$

Computation of T:

$$
\begin{aligned}
T(u) &= 6.090 \\
T(V) &= 1.163 \\
\hline
\log 4.927 &= 0.69258 \\
\log C &= 0.11858 \\
\log \sec \phi &= 0.00228 \\
\hline
\log T &= 0.81344 \qquad \therefore\ T = 6.51 \text{ seconds.}
\end{aligned}
$$

Computation of v:

$$\begin{aligned} \log u &= 2.85449 \\ \log \cos \phi &= 9.99809 \\ \log \sec \omega &= 0.00421 \\ \hline \log v &= 2.85679 \qquad \therefore\ v = 719 \text{ f. s.} \end{aligned}$$

For the shell, we have $d = 9.87$, $w = 107$, $\log C = 0.04075$, $X = 6000$ and $V = 1493$.

The results are as follows:

$$\begin{aligned} \phi &\doteq 5^\circ\, 9'; \\ \omega &= 8^\circ\, 19'; \\ T &= 6.48 \text{ seconds}; \\ v &= 691.8 \text{ f. s.} \end{aligned}$$

It will be seen from the above that the shot, though having a muzzle velocity 135 f. s. less than the shell, has a striking velocity, at 2000 yards from the gun, greater than the shell by 27 f. s. The time of flight and, therefore, the mean velocity are about the same for both projectiles. The shell has a less angle of departure and a greater angle of fall than the shot.

PROBLEM XIII.

Given the muzzle velocity (V) *and angle of departure* (ϕ), *to calculate the elements of the trajectory at the summit.*

Solution. At the summit we have (see equations (18) and (25))

$$m_0 = A = \frac{\sin 2\phi}{C}.$$

With the given value of V and that of m_0 computed as above, take from Table **m** the value of z_0; whence

$$x_0 = Cz_0.$$

Next, with the arguments V and z_0 take b_0 from Table **B**. We then have, by an obvious modification of (24) (since, at the summit, $m_0 = A$),

$$y_0 = \frac{b_0}{m_0} x_0 \tan \phi.$$

To compute the summit velocity, we have from (15) and the definition of A,

$$I(u_0) = A + I(V),$$

and from (7),

$$v_0 = u_0 \cos \phi.$$

To compute the time from the origin to the summit, we have

$$t_0 = \frac{C}{\cos \phi} \{T(u_0) - T(V)\}.$$

Example 1. Take the data of Ex. 2, Prob. IX, viz., $V = 1850$, $\phi = 10°$, $\log C = 0.70198$, and $A = m_0 = 0.06793$.

From Table **m** we have

$$z_0 = 2400 + \frac{100}{38}(679 - 647) = 2484.2;$$

and from Table **B**,

$$b_0 = 0.0360 + 0.842 \times 0.0023 = 0.0379.$$

$$\begin{aligned} \log z_0 &= 3.39519 \\ \log C &= 0.70198 \\ \hline \log x_0 &= 4.09717 \qquad \therefore\ x_0 = 12508 \text{ feet.} \end{aligned}$$

$$\begin{aligned} \log \tan \phi &= 9.24632 \\ \log b_0 &= 8.57864 \\ \text{a. c. } \log m_0 &= 1.16794 \\ \hline \log y_0 &= 3.09007 \qquad \therefore\ y_0 = 1230 \text{ feet.} \end{aligned}$$

$$\begin{aligned} A &= 0.06793 \\ I(V) &= 0.03727 \\ \hline I(u_0) &= 0.10520 \qquad \therefore\ u_0 = 1299.2 \end{aligned}$$

$$\begin{aligned} \log u_0 &= 3.11368 \\ \log \cos \phi &= 9.99335 \\ \hline \log v_0 &= 3.10703 \qquad \therefore\ v_0 = 1279.5 \end{aligned}$$

$$\begin{aligned} T(u_0) &= 2.903 \\ T(V) &= 1.291 \\ \hline & 1.612 \end{aligned}$$

$$\begin{aligned} \log 1.612 &= 0.20737 \\ \log C &= 0.70198 \\ \log \sec \phi &= 0.00665 \\ \hline t_0 &= 0.91600 \qquad \therefore\ t_0 = 8.24 \text{ seconds.} \end{aligned}$$

Example 2. With the data of Ex. 1, compute the co-ordinates of the summit when (*a*) $\phi = 5°$, and (*b*) when $\phi = 15°$.

(*a*) $\phi = 5°$:

$$\begin{aligned} \log \sin 2\phi &= 9.23967 \\ \log C &= 0.70198 \\ \hline \log m_0 &= 8.53769 \qquad \therefore\ m_0 = 0.0345 \end{aligned}$$

$$\therefore\ z_0 = 1400 + \frac{100}{28}(345 - 323) = 1478.6,$$

$$b_0 = 0.0172 + .786 \times .0016 = 0.0185.$$

$$\begin{aligned} \log z_0 &= 3.16985 \\ \log C &= 0.70198 \\ \hline \log x_0 &= 3.87183 \qquad \therefore\ x_0 = 7444.5 \text{ feet.} \\ \log \tan \phi &= 8.94195 \\ \log b_0 &= 8.26717 \\ \text{a. c. } \log m_0 &= 1.46231 \\ \hline \log y_0 &= 2.54326 \qquad \therefore\ y_0 = 349.3 \text{ feet.} \end{aligned}$$

(*b*) $\phi = 15°$:

$$\begin{aligned} \log \sin 2\phi &= 9.69897 \\ \log C &= 0.70198 \\ \hline \log m_0 &= 8.99699 \qquad \therefore\ m_0 = 0.09931 \end{aligned}$$

$$\therefore\ z_0 = 3200 + \frac{100}{47}(993 - 980) = 3227.7,$$

$$b_0 = 0.0563 + .277 \times .0029 = 0.0571.$$

$$\begin{aligned} \log z_0 &= 3.50889 \\ \log C &= 0.70198 \\ \hline \log x_0 &= 4.21087 \qquad \therefore\ x_0 = 16251 \text{ feet.} \\ \log \tan \phi &= 9.42805 \\ \log b_0 &= 8.75664 \\ \text{a. c. } \log m_0 &= 1.00301 \\ \hline \log y_0 &= 3.39857 \qquad \therefore\ y_0 = 2504 \text{ feet.} \end{aligned}$$

Effect of Altitude upon the Flight of a Projectile.—When the angle of departure is so great that the projectile reaches a high altitude, as in this last example, it is evident

that for a considerable portion of its trajectory the projectile meets with a less resistance than it would if it moved nearer the earth's surface; or, in other words, if the angle of departure were less. As our tables are based upon a resistance due to the density of the air at the level of the sea, the computed range of a projectile which moves in air of less density than this, and which, therefore, meets with less resistance than that contemplated by the tables, must be too short, and the computed angle of departure must be too great.

To remedy this, we assume a fictitious projectile of the same dimensions as the real projectile, but of sufficiently greater weight to traverse air of the *standard* density with the same facility as the real projectile traverses the rarefied air due to the mean height of the trajectory. With such a projectile it is evident that our tables would give correct results. A little consideration will show that the same object can be attained by multiplying the ballistic coefficient C by a suitable factor, greater than unity, depending upon the mean height of the trajectory, which latter we will designate by h. This factor should evidently be unity when $h =$ zero, and increase as h increases. Chauvenet gives to this factor the form

$$e^{\frac{h}{\lambda}},$$

in which e is the Naperian base and λ the height of the equivalent homogeneous atmosphere, supposed to be of uniform temperature, and which is taken at 27800 feet. The expression for C, when the correction for altitude is made, becomes, therefore,

$$C = \frac{\delta_{\prime}}{\delta}\frac{\omega}{cd^2}e^{\frac{h}{\lambda}}.$$

The values of $e^{\frac{h}{\lambda}}$ for every 100 feet of altitude from $h = 0$ to $h = 9900$ feet are given in the following table. To use it, look for the thousands in the first vertical column headed "h," and for the hundreds in the first horizontal column. At their inter-

section will be found the *decimal* part of the factor required, which must be annexed to unity, as in the second column.

TABLE OF ALTITUDE FACTORS.

h	0	100	200	300	400	500	600	700	800	900
0	1.0000	0036	0072	0108	0145	0181	0218	0255	0292	0329
1000	1.0366	0403	0441	0479	0516	0554	0592	0631	0669	0707
2000	1.0746	0785	0824	0863	0902	0941	0981	1020	1060	1100
3000	1.1140	1180	1220	1260	1301	0341	1382	1423	1464	1506
4000	1.1547	1589	1630	1672	1714	1756	1799	1841	1884	1927
5000	1.1970	2013	2057	2100	2144	2187	2231	2276	2320	2364
6000	1.2409	2454	2499	2544	2589	2634	2679	2725	2771	2817
7000	1.2863	2909	2956	3003	3049	3096	3144	3191	3239	3286
8000	1.3334	3382	3431	3479	3528	3576	3625	3675	3724	3773
9000	1.3823	3873	3923	3973	4023	4074	4125	4176	4227	4278

For direct fire the mean value of h, or mean height of the trajectory, is about two thirds the uncorrected maximum height, or height of summit. Therefore, to determine the value of h with which to enter the table, we have the following rule:

$$h = \tfrac{2}{3} y_0 .$$

It will be near enough in practice to take the *even hundred* nearest the computed value of h.

Example 3. Calculate the range and time of flight of Ex. 2, Prob. IX, making allowance for the height of the trajectory; (a) when $\phi = 10°$, and (b) when $\phi = 15°$.

(a) $\phi = 10°$. We have $y_0 = 1230$ feet (Ex. 1). Two thirds of this, to the nearest hundred, is 800 feet, which is the mean height of the trajectory.

$\therefore h = 800$; and from the table we find the altitude factor to be 1.0292, by which we must multiply the value of C heretofore used. We therefore have $\log C = 0.71448$. The further calculations are as follow:

$$\log \sin 2\phi = 9.53405$$
$$\log C = 0.71448$$
$$\log A = 8.81957 \qquad \therefore A = 0.0660$$

$$\therefore\ z = 4400 + \frac{100}{22}\{660 - 654\} = 4427.3. \quad \text{(Table } \mathbf{A.}\text{)}$$

$$\begin{aligned} \log z &= 3.64614 \\ \log C &= 0.71448 \\ \hline \log X &= 4.36062 \qquad \therefore\ X = 22941 \text{ feet.} \end{aligned}$$

$$\begin{aligned} z &= 4427.3 \\ S(V) &= 2916.9 \\ \hline S(V) &= 7344.2 \qquad \therefore\ u = 1031.2 \end{aligned}$$

$$\begin{aligned} T(u) &= 4.598 \\ T(V) &= 1.291 \\ \hline \log 3.307 &= 0.51943 \\ \log C &= 0.71448 \\ \log \sec \phi &= 0.00665 \\ \hline \log T &= 1.24056 \qquad \therefore\ T = 17.40 \text{ seconds.} \end{aligned}$$

The calculated range is, therefore, increased 224 feet when the diminished density of the air, due to the height of the trajectory, is taken into account—a distance too great to be neglected in accurate firing. The difference in the time of flight is very small, as was to be expected; since the increased range is compensated for by the greater mean velocity of the projectile. (See Prob. VIII.)

(*b*) $\phi = 15^\circ$. In this case we have $y_0 = 2504$ feet, $h = 1700$ feet, altitude factor $= 1.0631$ and $\log C = 0.72855$.

$$\begin{aligned} \log \sin 2\phi &= 9.69897 \\ \log C &= 0.72855 \\ \hline \log A &= 8.97042 \qquad \therefore\ A = 0.09342 \end{aligned}$$

$$\therefore\ z = 5500 + \frac{100}{26}(934 - 917) = 5565.4. \quad \text{(Table } \mathbf{A.}\text{)}$$

$$\log z = 3.74550$$
$$\log C = 0.72855$$
$$\log X = 4.47405 \qquad \therefore X = 29789 \text{ feet.}$$

Without using the altitude factor the computed range would have been 29128 feet; a difference of 661 feet. The difference between the computed times of flight in the two cases is $23.89 - 23.65 = 0.24$ seconds.

NOTE.—When ϕ does not exceed 5° the height of the trajectory has no material effect upon the range.

Example 4. Given $d = 24$ cm., $w = 215$ kg., $V = 529$ m.s. $= 1735.6$ f.s., $\phi = 12° 5'$ and $\delta = 1.275$ kg., to compute X. (Krupp, *Expériences de tir*, No. 56, page 4.)

$$\text{const. } \log = 1.28009 \quad \text{(See Ex. 3, Prob. VII.)}$$
$$\log w = 2.33244$$
$$\text{a. c. } \log d^2 = 7.23958$$
$$\text{a. c. } \log \delta = 9.89449$$
$$\log C = 0.74660$$
$$\log \sin 2\phi = 9.61214$$
$$\log m_0 = 8.86554 \qquad \therefore m_0 = 0.0734$$

$$\therefore z_0 = 2300 + \frac{100}{44}\{.712 \times 40 + 734 - 722\} = 2392.0.$$
(Table **m.**)

$$\therefore b_0 = 0.0400 + .92 \times .0026 - .712 \times .0022 = 0.0408.$$
(Table **B.**)

$$\log z_0 = 3.37876$$
$$\log C = 0.74660$$
$$\log x_0 = 4.12536 \qquad \therefore x_0 = 13346 \text{ feet.}$$
$$\log \tan \phi = 9.33057$$
$$\log b_0 = 8.61066$$
$$\text{a. c. } \log m_0 = 1.13446$$
$$\log y_0 = 3.20105 \qquad \therefore y_0 = 1589 \text{ feet.}$$

We have, therefore, $h = 1100$ feet, and the altitude factor $= 1.0403$. Whence $\log C = 0.76376$. We are now prepared to compute the range by Prob. IX.

$$\begin{aligned} \log \sin 2\phi &= 9.61214 \\ \log C &= 0.76376 \\ \hline \log A &= 8.84838 \qquad \therefore A = 0.0705 \end{aligned}$$

$$\therefore z = 4200 + \frac{100}{25}\{.712 \times 39 + 705 - 719\} = 4255.1.$$

(Table **A.**)

$$\begin{aligned} \log z &= 3.62891 \\ \log C &= 0.76376 \\ \hline \log X &= 4.39267 \qquad \therefore X = 24699 \text{ feet.} \end{aligned}$$

$$\begin{aligned} \text{Mean measured range} &= 24935 \text{ “} \\ \hline \text{Difference} &= 236 \text{ feet.} \end{aligned}$$

To determine the angle of fall, we have, from Table **B**, using the arguments z and V,

$$B = 0.1025 + .551 \times .0040 - .712 \times .0046 = 0.1014.$$

$$\begin{aligned} \log \tan \phi &= 9.33057 \\ \log B &= 9.00604 \\ \text{a. c.} \log A &= 1.15162 \\ \hline \log \tan \omega &= 9.48823 \qquad \therefore \omega = 17^\circ\, 6' \end{aligned}$$

For the striking velocity we have

$$\begin{aligned} z &= 4255.1 \\ S(V) &= 3366.2 \\ \hline S(u) &= 7621.3 \qquad \therefore u = 1007.7 \end{aligned}$$

$$\begin{aligned} \log u &= 3.00333 \\ \log \cos \phi &= 9.99027 \\ \log \sec \omega &= 0.01964 \\ \hline \log v &= 3.01324 \qquad \therefore v = 1031 \text{ f. s.} \end{aligned}$$

Time of flight:

$$T(u) = 4.871$$
$$T(V) = 1.542$$

$$\log 3.329 = 0.52231$$
$$\log C = 0.76376$$
$$\log \sec \phi = 0.00973$$

$$\log T = 1.29580 \qquad \therefore\ T = 19.76 \text{ seconds.}$$

Example 5. Given $d = 40$ cm., $w = 920$ kg., $\delta = 1.206$ kg., $c = 0.9$, $\log C = 0.95841$, $V = 1804.5$ f. s. and ϕ 18°, to compute the trajectory.

$$\log \sin 2\phi = 9.76922$$
$$\log C = 0.95841$$

$$\log m_0 = 8.81081 \qquad \therefore\ m_0 = 0.0647$$

$$\therefore\ z = 2300 + \frac{100}{39}\{.09 \times 34 + 647 - 644\} = 2315.5.$$

(Table **III**.)

$$\therefore\ b_0 = 0.0357 + .155 \times .0023 - .09 \times .0019 = 0.0359.$$

(Table **B**.)

$$\log z_0 = 3.36464$$
$$\log C = 0.95841$$

$$\log x_0 = 4.32305$$
$$\log \tan \phi = 9.51178$$
$$\log b_0 = 8.55509$$
$$\text{a. c. } \log m_0 = 1.18916$$

$$\log y_0 = 3.57908 \qquad \therefore\ y_0 = 3794 \text{ feet.}$$

We have, therefore, $h = 2500$ feet, and the altitude factor $= 1.0941$. Whence $\log C = 0.99747$.

$$\log \sin 2\phi = 9.76922$$
$$\log C = 0.99747$$

$$\log A = 8.77175 \qquad \therefore\ A = 0.05912$$

$$\therefore z = 3900 + \frac{100}{21}\{.09 \times 31 + 591 - 579\} = 3970.4. \quad \text{(Table } \mathbf{A}.)$$

$$\log z = 3.59883$$
$$\log C = 0.99747$$
$$\log X = 4.59630 \quad \therefore X = 39473 \text{ feet.}$$
$$= 12031 \text{ metres.}$$
$$= 7.476 \text{ miles.}$$

The mean of eight shots fired with this gun at Meppen, April 29, 1886, with 18° elevation, was 39808 feet. The calculated range is, therefore, short of the actual mean range by only 335 feet; and this difference can be accounted for by the jump of the gun.

To determine the jump which will cause the calculated range to agree with the measured range in this example, we proceed as follows:

We have $X = 39808$ feet (mean range), $V = 1804.5$ f. s., and $\log C = 0.99747$, to calculate ϕ, by Prob. XII,

$$\log X = 4.59997$$
$$\log C = 0.99747$$
$$\log z\ 3.60250 \qquad \therefore z = 4004.1$$

$$\therefore A = 0.0600 + .041 \times .0022 - .09 \times .0032 = 0.0598. \quad \text{(Table } \mathbf{A}.)$$

$$\log A = 8.77670$$
$$\log C = 0.99747$$
$$\log \sin 2\phi = 9.77417 \qquad \therefore \phi = 18^\circ\ 14'$$

The calculated jump is therefore 14′, which is certainly a very close approximation to the actual jump. Compare Ex. 5, Prob. IX.

A jump of 11′ will, in like manner, make the computed range agree with the actual mean range, in Ex. 4.

We will complete Ex. 5, by computing the angle of fall, etc., using for this purpose $V = 1804.5$, $z = 3970.4$, and $\log C = 0.99747$.

Angle of fall.

$$B = 0.0824 + .704 \times .0036 - .09 \times .0041 = 0.08456. \text{ (Table } \mathbf{B}.)$$

$$\begin{aligned} \log \tan \phi &= 9.51178 \\ \log B &= 8.92717 \\ \text{a. c. } \log A &= 1.22825 \\ \hline \log \tan \omega &= 9.66720 \quad \therefore \ \omega = 24^\circ\ 56' \end{aligned}$$

Striking velocity.

$$\begin{aligned} z &= 3970.4 \\ S(V) &= 3092.3 \\ \hline S(u) &= 7062.7 \quad \therefore \ u = 1058.1 \end{aligned}$$

$$\begin{aligned} \log u &= 3.02453 \\ \log \cos \phi &= 9.97821 \\ \log \sec \omega &= 0.04249 \\ \hline \log v &= 3.04523 \quad \therefore \ v = 1109.8 \text{ f. s.} \end{aligned}$$

Time of flight.

$$\begin{aligned} T(u) &= 4.331 \\ T(V) &= 1.387 \\ \hline \end{aligned}$$

$$\begin{aligned} \log 2.944 &= 0.46894 \\ \log C &= 0.99747 \\ \log \sec \phi &= 0.02179 \\ \hline \log T &= 1.48820 \quad \therefore \ T = 30.8 \text{ seconds.} \end{aligned}$$

Penetration of Armor.—It may be of interest to know the number of inches of wrought-iron armor which a projectile

fired from this gun (the most powerful yet constructed) will penetrate at the extreme range of seven and one-half miles. This may be computed by Maitland's formula for penetration. (See p. 26.)

But to use it in the form there given, it is necessary to reduce the diameter of the projectile to inches and its weight to pounds. We may avoid this labor by introducing the proper units into the formula, which thus becomes

$$\tau = \frac{v}{257.065}\left(\frac{w}{d}\right)^{\frac{1}{2}} - 0.055d,$$

in which v is in feet per second, w in kilogrammes, d in centimetres, and τ in inches.

Should v be given in metres per second, the other units remaining as above, the formula would become

$$\tau = \frac{v}{78.352}\left(\frac{w}{d}\right)^{\frac{1}{2}} - 0.055d.$$

The computation is as follows:

$$\begin{array}{rl}
\log w = & 2.96379 \\
\log d = & 1.60206 \\
\hline
2) & 1.36173 \\
\hline
 & 0.68086 \\
\hline
\log v = & 3.04523 \\
\text{a. c. } \log 257.065 = & 7.58996 \\
\hline
\log 20.70 = & 1.31605 \\
0.055d = & 2.20 \\
\hline
\tau = & 18.50 \text{ inches.}
\end{array}$$

PROBLEM XIV.

Given the muzzle velocity (V), *to determine the angle of departure* (ϕ) *which will cause a projectile to hit an object situated above or below the level of the gun; also the striking angle* (θ), *the striking velocity* (v), *and the time of flight* (t).

SOLUTION. (FIRST METHOD.)

Let x and y be the co-ordinates of the given object (see page 3) and s its distance from the gun, whence

$$s = \sqrt{x^2 + y^2} \quad \text{and} \quad x = \sqrt{(s+y)(s-y)}.$$

Also, let ϵ be the angular distance of the object above (or below) the level of the gun, and therefore

$$\tan \epsilon = \frac{y}{x}.$$

If the object is *above* the level of the gun, y and ϵ are positive; while if it is *below* the level of the gun they are both negative.

Compute z by the formula

$$z = \frac{s}{C},$$

and with the arguments V and z take a from Table **A.** Then from (22) we have

$$\frac{y}{x} = \tan \epsilon = \tan \phi \left\{ 1 - \frac{aC}{\sin 2\phi} \right\}.$$

Solving with reference to $\tan \phi$, we have

$$\tan \phi = \frac{1}{aC} \left\{ 1 - \sqrt{1 - aC(aC + 2 \tan \epsilon)} \right\},$$

from which to compute ϕ.

To compute θ we take m from Table **m** with the arguments V and z; and then (Eq. 23)

$$\tan \theta = \tan \phi \left\{ 1 - \frac{mC}{\sin 2\phi} \right\}.$$

For the striking velocity we have

$$S(u) = z + S(V),$$

and

$$v = u \frac{\cos \phi}{\cos \theta}.$$

The time of flight is given by the equation

$$t = \frac{C}{\cos \phi} \left\{ T(u) - T(V) \right\}.$$

Example 1. "An enemy's ship attacking Gibraltar, confines itself, at a range of 3000 yards, to firing at the Signal Station, which is known to be 1270 feet high. The ship is using a 12-inch gun of 45 tons, with a 295-pound charge, giving a muzzle velocity of 1910 f. s. Find the necessary elevation." (Prof. Greenhill, in *Proceedings Royal Artillery Institution*, No. 14, Volume XV.)

Here $V = 1910$, $w = 714$, $d = 12$, $y = 1270$, $s = 9000$, and (taking the coefficient of reduction $c = 0.95$) log $C = 0.71762$.

Computation of ϕ:

$$\begin{aligned}
\log (s + y) &= 4.01157 \\
\log (s - y) &= 3.88818 \\
\hline
2)\ &7.89975 \\
\log x &= 3.94988 \\
\log y &= 3.10380 \\
\hline
\log \tan \epsilon &= 9.15392 \qquad \therefore \epsilon = 8^\circ\ 6'\ 43'' \\
\therefore \tan \epsilon &= 0.142535
\end{aligned}$$

$$\log s = 3.95424$$
$$\log C = 0.71762$$
$$\log z = 3.23662 \qquad \therefore z = 1724.3$$

$\therefore a = 0.0179 + .243 \times .0013 - .2 \times .0009 = 0.0180$. (Table **A**.)

$$\log a = 8.25527$$
$$\log C = 0.71762$$
$$\log 0.09395 = 8.97289 = \log aC$$
$$2 \tan \epsilon = 0.28507$$
$$\log 0.37902 = 9.57866$$
$$\text{(Sub. from 1)} \log 0.03561 = 8.55155$$
$$\log 0.96439 = 9.98425 \qquad \text{(Divide log by 2)}$$
$$\text{(Sub. from 1)} \log 0.98204 = 9.99213$$
$$\log 0.01796 = 8.25431$$
$$\log aC = 8.97289$$
$$\log \tan \phi = 9.28142 \qquad \therefore \phi = 10^\circ\, 49'\, 22''$$

The angle of elevation of the gun above the object is, therefore, $10^\circ\, 49'\, 22'' - 8^\circ\, 6'\, 43'' = 2^\circ\, 42'\, 39''$.

Computation of θ:

We have, from Table **m**,

$$m = 0.0389 + .243 \times .0030 - .2 \times .0019 = 0.0392.$$

$$\log m = 8.59329$$
$$\log C = 0.71762$$
$$\text{a. c.} \log \sin 2\phi = 0.43273$$
$$\text{(Sub. from 1)} \log 0.55417 = 9.74364$$
$$\log 0.44583 = 9.64917$$
$$\log \tan \phi = 9.28142$$
$$\log \tan \theta = 8.93059 \qquad \therefore \theta = 4^\circ\, 52'$$

Computation of v:

$$\begin{aligned} z &= 1724.3 \\ S(V) &= 2692.2 \\ \hline S(u) &= 4416.5 \qquad \therefore u = \mathbf{1495.0} \end{aligned}$$

$$\begin{aligned} \log u &= 3.17464 \\ \log \cos \phi &= 9.99219 \\ \log \sec \theta &= 0.00157 \\ \hline \log v &= 3.16840 \qquad \therefore v = \mathbf{1474\ f.\ s.} \end{aligned}$$

Computation of t:

$$\begin{aligned} T(u) &= 2.194 \\ T(V) &= 1.171 \\ \hline \log 1.023 &= 0.00988 \\ \log C &= 0.71762 \\ \log \sec \phi &= 0.00781 \\ \hline \log t &= 0.73531 \qquad \therefore t = 5.44 \text{ seconds.} \end{aligned}$$

Horizontal Range.—We will now compute the angle of departure (ϕ) for a *horizontal range* of 3000 yards, with the gun of this Example. Our data will be the same as before, except that $y = 0$, and $s = X = 9000$. z will remain the same and a will become A. Therefore

$$\sin 2\phi = AC = aC.$$

But we have already found

$$\log aC = 8.97289 = \log \sin 2\phi;$$

$$\therefore \phi = 2^\circ\ 41'\ 43''.$$

Comparing this value of ϕ with the angle of elevation of the gun above the Signal Station as deduced in the above Example, it will be seen that by revolving the *horizontal trajectory* through a *positive* vertical angle of $8^\circ\ 6'\ 43''$ (the distance to the

object remaining the same), the angle of elevation *with reference to the object aimed at* is slightly increased, owing to the variation in the direction of gravity with reference to the direction of motion of the projectile, which in this case diminishes the velocity of the latter and, therefore, necessitates an increase in the angle of elevation in order to reach the object. The variation of the angle of elevation is, however, less than one minute in our example, and in practice might be taken at $2° 42'$ in both cases.

Example 2. Suppose the gun of Ex. 1 to be fired *from the Signal Station* at a ship whose distance is 3000 yards. Find the values of ϕ, θ, v, and t.

The only change in the data of Ex. 1 is the sign of y, which becomes minus; and, therefore tan, ϵ and ϵ are both negative.

The computations are as follow:

$$aC = \quad 0.09395$$
$$2 \tan \epsilon = -0.28508$$

$$\log - 0.19113 = 9.28133_n{}^*$$
$$\log aC = 8.97289$$

$$\text{(Sub. from 1)} \log - 0.01796 = 8.25422_n$$

$$\log 1.01796 = 0.00773 \qquad \text{(Divide log by 2.)}$$
$$\text{(Sub. from 1)} \log 1.00893 = 0.00386$$

$$\log - 0.00893 = 7.95085_n$$
$$\log aC = 8.97289$$

$$\log \tan \phi = 8.97796_n \quad \therefore \phi = -5° 25' 47''$$

As the angle of depression of the ship is $- 8° 6' 43''$, the angle of elevation of the gun above the ship is $2° 40' 56''$. Therefore, by revolving the horizontal trajectory through a *negative* angle of $8° 6' 43''$, the angle of elevation is slightly diminished, since in this case the variation in the direction of

* The subscript n annexed to a logarithm indicates that the corresponding number is negative.

gravity with reference to the direction of motion of the projectile increases the velocity of the latter.

Computation of θ:

$$\begin{aligned}
\log mC &= 9.31091 \\
\log \sin 2\phi &= 9.27508_n \\
\text{(Sub. from 1) } \log - 1.08600 &= 0.03583_n \\
\log 2.08600 &= 0.31931 \\
\log \tan \phi &= 8.97796_n \\
\log \tan \theta &= 9.29727_n \qquad \therefore \theta = -11^\circ\ 13'
\end{aligned}$$

Computation of v:

[It will be observed that u is the same in both examples.]

$$\begin{aligned}
\log u &= 3.17464 \\
\log \cos \phi &= 9.99804 \\
\log \sec \theta &= 0.00838 \\
\log v &= 3.18106 \qquad \therefore v = 1517 \text{ f. s.}
\end{aligned}$$

Computation of t:

$$\begin{aligned}
\log C\{T(u) - T(V)\} &= 0.72750 \\
\log \cos \phi &= 9.99804 \\
\log t &= 0.72946 \qquad \therefore t = 5.36 \text{ seconds.}
\end{aligned}$$

The striking velocity in Ex. 1 is 1474 f. s.; and in Ex. 2 it is 1517 f. s. The difference is due to the action of gravity, which impedes the motion of the projectile in the one case, and assists it in the other.

Rigidity of the Trajectory.—The two preceding examples illustrate an important principle known as the Rigidity of the Trajectory,* which assumes that the relations existing between the elements of a trajectory and the chord represent-

* Called by German writers *das Schwenken der Bahnen*, and by the French *l'hypothèse de la rigidité de la trajectoire.*

ing the range, are sensibly the same whether the latter be horizontal or inclined to the horizon, *within certain limits.*

This principle gives the following simple rule for determining the angle of departure when the object aimed at is above, or below, the level of the gun:

Calculate the angle of departure for a horizontal range equal to the distance of the object from the gun, and add to it the angle of elevation (or depression) of the object; which gives the angle of departure sought.

Example 3. According to the range table the 8-inch M. L. rifle (converted) requires an angle of departure of $5° 43'$ for a range of 3000 yards. What would be the angle of departure supposing the gun to be 40 feet higher than the object aimed at?

Here s and x differ insensibly from each other and $= 9000$ feet; $y = -40$ feet.

$$\therefore \tan \epsilon = -\frac{40}{9000}$$

$$\log 40 = 1.60206_n$$
$$\log 9000 = 3.95424$$

$$\log \tan \epsilon = 7.64782_n \qquad \therefore \epsilon = -15'$$

$$\therefore \phi = 5° 43' + (-15') = 5° 28'$$

The above rule is applicable to *all* our sea-coast guns, which are but moderately elevated above the level of the sea; and we have also shown that it is sufficiently accurate for *high-powered guns* even in the extreme case of the highest battery at Gibraltar. But with guns of less power, giving trajectories of considerable curvature, the angle of departure computed by the rule, for the Signal Station at Gibraltar, would be wrong by some minutes. This is illustrated by the following example.

Example 4. "It was recently necessary to fire a 64-pounder converted gun with a charge of 8 pounds, giving a muzzle velocity of 1260 f. s., from the Signal Station at Gibraltar, 1270

feet above the level of the sea. The object fired at was 2000 yards from the muzzle of the gun."

Find the angle of departure.

Here $d = 6.3$, $w = 64.5$, $V = 1260$, $s = 6000$, $y = -1270$, $c = 1$ and $\log C = 0.21088$.

We will first compute ϕ by the method of Example 1.

$$\log(s+y) = 3.67486$$
$$\log(s-y) = 3.86153$$
$$2)\ 7.53639$$
$$\log x = 3.76820$$
$$\log y = 3.10380_n$$
$$\log \tan \epsilon = 9.33560_n \qquad \therefore\ \epsilon = -12^\circ\ 13'$$
$$\therefore\ \tan \epsilon = -0.21657$$
$$\log s = 3.77815$$
$$\log C = 0.21088$$
$$\log z = 3.56727 \qquad \therefore\ z = 3692.1$$

$\therefore\ a = .0989 + .921 \times .0034 - .2 \times .0057 = 0.1009$. (Table **A.**)

$$\log a = 9.00389$$
$$\log C = 0.21088$$
$$\log 0.16397 = 9.21477 = \log aC$$
$$2 \tan \epsilon = -0.43314$$
$$\log -0.26917 = 9.43003_n$$
$$\log -0.04414 = 8.64480_n$$
$$\log 1.04414 = 0.01876$$
$$\log 1.02183 = 0.00938$$
$$\log -0.02183 = 8.33905_n$$
$$\log aC = 9.21477$$
$$\log \tan \phi = 9.12428_n$$

Therefore the angle of departure is $-7^\circ 35'$.

For a *horizontal range* of 6000 feet, we have for the angle of departure,

$$\sin 2\phi = AC$$
$$\therefore \log \sin 2\phi = 9.21477$$
$$\therefore \phi = 4^\circ 43'$$

Therefore by the rule given above, the angle of departure in this example would be

$$4^\circ 43' + (-12^\circ 13') = -7^\circ 30',$$

differing by $5'$ from its true value.

Continuing the calculations, we find

$$\theta = -17^\circ 46'$$
$$v = 929 \text{ f. s.}$$
$$t = 5.88 \text{ seconds.}$$

Example 5. Suppose the gun of Example 4 to be fired *at the Signal Station* from the level of the sea, the other data remaining the same. Calculate ϕ, θ, v, and t.

$$\text{Answers: } \phi = 17^\circ 2'$$
$$\theta = 6^\circ 2'$$
$$v = 858 \text{ f. s.}$$
$$t = 6.09 \text{ seconds.}$$

By the rule we find

$$\phi = 4^\circ 43' + 12^\circ 13' = 16^\circ 56',$$

differing by $6'$ from its true value.

SECOND METHOD.

When the value of the auxiliary quantities a and m cannot be taken from the tables, they must be computed by the equations

$$S(u) = z + S(V),$$

$$a = \frac{A(u) - A(V)}{S(u) - S(V)} - I(V),$$

and

$$m = I(u) - I(V).$$

The remaining calculations are the same as by the first method.

Example 6. A 15-inch S. B. gun, mounted upon a bluff overlooking the sea, fires a plunging shot at the deck of a ship distant 1000 yards and 300 feet below the level of the gun. Suppose the muzzle velocity to be 1700 f. s., and weight of solid shot 450 pounds, what must be the angle of depression, and what the racking energy of projectile on striking?

Here $\log C = 0.30859$ (see Ex. 1, Prob. 1), $s = 3000$, and $y = -300$.

$$\begin{aligned}
\log s &= 3.47712 \\
\log C &= 0.30859 \\
\hline
\log z &= 3.16853
\end{aligned}$$

$$\begin{aligned}
z &= 1474 \\
S(V) &= 798 \\
\hline
S(u) &= 2272 \qquad \therefore u = 1259.2
\end{aligned}$$

$$\begin{aligned}
A(u) &= 57.85 \\
A(V) &= 5.73 \\
\hline
\end{aligned}$$

$$\begin{aligned}
\log 52.12 &= 1.71700 \\
\log z &= 3.16853 \\
\hline
\log 0.03536 &= 8.54847 \\
I(V) = 0.01517 & \\
\hline
\log 0.02019 &= 8.30514 = \log a \\
\log C &= 0.30859 \\
\hline
\log 0.04109 &= 8.61373 = \log aC
\end{aligned}$$

$$\begin{array}{r}
\log(s+y) = 3.43136 \\
\log(s-y) = 3.51851 \\
\hline
2)\,6.94987 \\
\hline
\end{array}$$

$$\log x = 3.47494 \qquad \therefore x = 2985 \text{ feet.}$$

$$\log y = 2.47712_n$$

$$\log \tan \epsilon = 9.09218_n \qquad \therefore \epsilon = -5^\circ\, 44'\, 20''$$

$$\begin{array}{rl}
\therefore 2 \tan \epsilon = & -0.20101 \\
aC = & 0.04109 \\
\hline
\end{array}$$

$$\log -0.15992 = 9.20390_n$$

$$\log aC = 8.61373$$

$$\log -0.006571 = 7.81763_n$$

$$\log 1.006571 = 0.00284$$

$$\log 1.00328 = 0.00142$$

$$\log -0.00328 = 7.51587_n$$

$$\log aC = 8.61373$$

$$\log \tan \phi = 8.90214_n \qquad \therefore \phi = -4^\circ\, 33'\, 50''$$

Computation of θ.

$$I(u) = 0.06010$$

$$I(V) = 0.01517$$

$$\log 0.04493 = 8.65254 = \log m$$

$$\log C = 0.30859$$

$$\log \operatorname{cosec} 2\phi = 0.79960_n$$

$$\log -0.57641 = 9.76073_n$$

$$\log 1.57641 = 0.19767$$

$$\log \tan \phi = 8.90214_n$$

$$\log \tan \theta = 9.09981_n \qquad \therefore \theta = -7^\circ\, 10'\, 20''$$

Computation of Striking Energy:

$$\begin{aligned} \log u &= 3.10009 \\ \log \cos \phi &= 9.99862 \\ \log \sec \theta &= 0.00341 \\ \hline \log v &= 3.10212 \qquad \therefore v = 1265.1 \text{ f. s.} \\ 2 \log v &= 6.20424 \\ \text{Const. } \log &= 7.49462 \qquad \text{(See page 23.)} \\ \hline \log E &= 3.69886 \qquad \therefore E = 4998.8 \text{ foot-tons.} \end{aligned}$$

The angle of depression in this case could have been computed with all desired accuracy by using the rule given on page 102, as follows:

We have $\log aC = 8.61373$; and therefore

$$\begin{aligned} \phi &= \quad 1^\circ\ 10'\ 39'' \\ \epsilon &= -5\ \ 44\ \ 20 \\ \hline \end{aligned}$$

Angle of depression $= -4^\circ\ 33'\ 41''$—which differs but 9 seconds from the former value—a quantity too small to be noticed.

Example 7. A 12-pound shrapnel fired from the 3-inch M. L. rifle has a muzzle velocity of 862 f. s. With what elevation should it be fired so as to burst 15 feet above and 50 yards in front of a target at a distance of 578 yards?

Here $d = 3$, $w = 12$, $c = 1$, $C = \frac{4}{3}$, $V = 862$, $y = 15$, and $x = s = 3\,(578 - 50) = 1584$.

$$\begin{aligned} z = \tfrac{3}{4} \times 1584 &= \ \ 1188.0 \\ S(V) &= \ \ 9850.5 \\ \hline S(u) &= 11038.5 \qquad \therefore u = v = 800.46 \text{ f. s.} \end{aligned}$$

$$
\begin{array}{rl}
A(u) = & 1745.14 \\
A(V) = & 1224.02 \\
\hline
\log 521.12 = & 2.71694 \\
\log z = & 3.07482 \\
\hline
\log 0.43866 = & 9.64212 \\
I(V) = & 0.38450 \\
\hline
\log 0.05416 = & 8.73368 = \log a \\
\log C = & 0.12494 \\
\hline
\log \sin 2\phi = & 8.85862 \qquad \therefore \phi = 2^\circ\, 4'\, 14'' \\
\log y = & 1.17609 \\
\log x = & 3.19976 \\
\hline
\log \tan \epsilon = & 7.97633 \qquad \therefore \epsilon = 32'\, 33''
\end{array}
$$

$\therefore$ Angle of departure $= 2^\circ\, 4'\, 14'' + 32'\, 33'' = 2^\circ\, 36'\, 47''$

To determine the *angle of elevation* we should have to subtract the jump, which in this piece varies from 20′ to 30′.

PROBLEM XV.

To deduce a formula for computing ordinates to a given trajectory.

SOLUTION. (FIRST METHOD.)

From (24) we have

$$y = \frac{\tan \phi}{A} \{A - a\} x,$$

by means of which y can be readily computed for given values of x.

As the trajectory is supposed to be given, V, X, and ϕ are either known or can be computed by methods already considered; and then A can either be taken from Table **A** with the proper arguments, or it can be computed by the equation

$$A = \frac{\sin 2\phi}{C}.$$

The quantity a varies with x, and must be taken from Table **A** for each assumed value of x, with the arguments V and $\frac{x}{C}$, or z.

If the object be to plot the trajectory by means of rectangular co-ordinates, the assumed values of x should have a constant difference. The co-ordinates of the summit should also be computed by the method of Prob. XIII.

Example 1. Compute ordinates 2000 feet apart for the trajectory of the 8-inch B. L. rifle, with the following data: $V = 1850$ f. s., $\phi = 6° 43'$, and $\log C = 0.71965$.

We will first compute the range and maximum ordinate by Probs. IX and XIII, as follows:

$$\begin{aligned} \log \sin 2\phi &= 9.36608 \\ \log C &= 0.71965 \\ \hline \log A &= 8.64643 \qquad \therefore A = 0.04430 \end{aligned}$$

$$\therefore z = 3300 + \tfrac{100}{18}(443 - 435) = 3344.4 \text{ (Table } \mathbf{A}).$$

$$\begin{aligned} \log z &= 3.52432 \\ \log C &= 0.71965 \\ \hline \log X &= 4.24397 \qquad \therefore X = 17538 \text{ feet.} \end{aligned}$$

For the maximum ordinate we have

$$m_0 = A = 0.0443.$$

Therefore from Table **m** we find

$$z_0 = 1800 + \tfrac{100}{32}(443 - 441) = 1806.35;$$

and from Table **B**,

$$b_0 = 0.0239 + .0635 \times .0019 = 0.0240.$$

$$\begin{aligned} \log z_0 &= 3.25680 \\ \log C &= 0.71965 \\ \hline \log x_0 &= 3.97645 \qquad \therefore x_0 = 9472 \text{ feet.} \\ \log \tan \phi &= 9.07103 \\ \log b_0 &= 8.38021 \\ \text{a. c. } \log m_0 &= 1.35357 \\ \hline \log y_0 &= 2.78126 \qquad \therefore y_0 = 604.3 \text{ feet.} \end{aligned}$$

The general expression for y, by applying the numbers found above, becomes

$$y = 2.6583(0.0443 - a)x;$$

and the remainder of the work consists in computing $\frac{x}{C}$ (or z)

for each given value of x, and taking from Table **A** the corresponding values of a, which must be substituted successively in the above equation.

The work may be tabulated to advantage as follows:

x feet.	z	a	$A - a$	y feet.	θ
0	0.0	0.0000	0.0443	0	$+6° 43'$
2000	381.4	.0037	.0406	216	$5° 35'$
4000	762.8	.0077	.0366	389	$4° 19'$
6000	1144.2	.0120	.0323	515	$2° 52'$
8000	1525.6	.0166	.0277	589	$+1° 17'$
9472	1806.3	.0203	.0240	604	$0° 00'$
10000	1907.0	.0216	.0227	603	$-0° 29'$
12000	2288.4	.0270	.0173	552	$2° 29'$
14000	2669.8	.0328	.0115	428	$4° 41'$
16000	3051.3	.0392	.0051	217	$7° 7'$
17538	3344.4	0.0443	0.0000	0	$-9° 8'$

The numbers in the first column are the values of x for which ordinates are to be computed, including x_0 and X. The second column contains the values of z obtained by dividing the numbers in the first column by C. The values of a in the third column are taken from Table **A**, with V and z as the arguments; and these subtracted from A are placed in the fourth column. Finally the products of the numbers on the same line in the first and fourth columns with the constant multiplier 2.6583, give the values of y in the fifth column.

The *inclination* of the trajectory at the points whose co-ordinates have been determined, can be computed by Eq. (25), which becomes in this case

$$\tan \theta = 2.6583(0.0443 - m).$$

The results are given in the last column.

By means of these values of x, y, and θ, the trajectory can be easily and accurately plotted.

Example 2. "Firing on the same level as the target, with the 12-pounder B. L. gun at 2000 yards range, it is required to know at what height a shrapnel shell will be if burst 200, 150,

100, and 50 yards short; also the angle of descent and time of flight for each." *

Here $d = 3$, $w = 12.31$, $V = 1710$, $X = 6000$, $c = 0.87$, and $\log C = 0.19650$, to calculate y, θ, and t, when $x = 5400$, 5550 5700, and 5850 feet, respectively.

We will first compute ϕ, ω, and T for the given range (6000 feet) by Prob. XII.

$$\begin{aligned} \log X &= 3.77815 \\ \log C &= 0.19650 \\ \hline \log z &= 3.58165 \qquad \therefore z = 3816.4 \end{aligned}$$

$$\therefore A = 0.0620 \text{ and } B = 0.0869$$

$$\begin{aligned} \log A &= 8.79239 \\ \log C &= 0.19650 \\ \hline \log \sin 2\phi &= 8.98889 \qquad \therefore \phi = 2^\circ\ 47'\ 50'' \end{aligned}$$

$$\begin{aligned} \log \tan \phi &= 8.68895 \\ \log B &= 8.93902 \\ \text{a. c. } \log A &= 1.20761 \\ \hline \log \tan \omega &= 8.83558 \qquad \therefore \omega = 3^\circ\ 55'\ 4'' \end{aligned}$$

$$\begin{aligned} z &= 3816.4 \\ S(V) &= 3470.8 \\ \hline S(u) &= 7287.2 \qquad \therefore u = v = 1036.4 \text{ f. s.} \end{aligned}$$

$$\begin{aligned} T(u) &= 4.543 \\ T(V) &= 1.602 \\ \hline \end{aligned}$$

$$\begin{aligned} \log 2.941 &= 0.46849 \\ \log C &= 0.19650 \\ \hline \log T &= 0.66499 \qquad \therefore T = 4.62 \text{ seconds.} \end{aligned}$$

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We next find the general expressions for y and θ (by applying numbers already found) to become .

$$y = 0.78807\,(0.0620 - a)x,$$

and

$$\tan\theta = 0.78807\,(0.0620 - m).$$

The values of a and m must be taken from Tables **A** and **m** with the arguments $V = 1710$ and $z = \frac{x}{C}$. We also calculate the value of u for each value of x by the equation

$$S(u) = z + S(V)$$

and then t by the formula

$$t = S(T(u) - T(V)).$$

The results are given in the following table:

$\log C = 0.19650$; $V = 1710$; $S(V) = 3470.8$; $T(V) = 1.602$.

x feet.	z	$u = v$	a	m	y feet.	θ	t
5400	3434.7	1074.8	0.0537	0.1270	35.3	$-2°\ 56'$	4.07
5550	3530.2	1064.5	.0557	.1323	27.6	3 10	4.20
5700	3625.6	1054.7	.0578	.1377	18.9	3 25	4.34
5850	3721.0	1045.3	.0599	.1432	9.7	3 40	4.48

SECOND METHOD.

When the values of A, a, and m are not obtainable from the tables, they may be computed as in the second method of Prob. XIV. The following is, however, preferable: Multiplying Eq. (3) by Eq. (1), and reducing by Eq. (6), we have

$$y = \frac{C^2}{2\cos^2\phi}\{I(u_0)z + A(V) - A(u)\}.$$

In connection with this equation we use the following when V and ϕ are given:

$$z = \frac{x}{C},$$

$$S(u) = z + S(V),$$

and

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V).$$

If ϕ is not given it will be necessary to compute it from V and X, as explained in the Second Method of Prob. XII.

Example 3. Compute ordinates 100 yards apart for the 1000-yard trajectory of the Springfield rifle. Also the co-ordinates of the summit.

Here $d = 0.45$, $w = 500$ grains $= \frac{1}{14}$ pound, $X = 3000$, $c = 1$, $V = 1301$ and $\log C = 9.54745$.

First compute $I(u_0)$ and ϕ, and then the factor $\frac{C^2}{2\cos^2\phi}$, as follows:

$$\begin{array}{rl} \log X = & 3.47712 \\ \log C = & 9.54745 \\ \hline \log z = & 3.92967 \\ \\ \therefore z = & 8505.0 \\ S(V) = & 5393.8 \\ \hline S(u) = & 13898.8 \qquad \therefore u = 676.17 \end{array}$$

$$\begin{array}{rl} A(u) = & 3623.32 \\ A(V) = & 213.15 \\ \hline \end{array}$$

$$\begin{array}{rl} \log 3410.17 = & 3.53278 \\ \log z = & 3.92667 \\ \hline \log 0.40097 = & 9.60311 \qquad = \log I(u_0) \end{array}$$

$$\begin{array}{rl} I(V) = & 0.10483 \\ \hline & 0.29614 \end{array}$$

$$
\begin{aligned}
\log 0.29614 &= 9.47150 \\
\log C &= 9.54745 \\
\hline
\log \sin 2\phi &= 9.01895 \qquad \therefore\ \phi = 2^\circ\ 59'\ 53'' \\
2 \log C &= 9.09490 \\
\log 0.5 &= 9.69897 \\
2 \log \sec \phi &= 0.00119 \\
\hline
\log 0.062383 &= 8.79506 \qquad = \log \frac{C^2}{2 \cos^2 \phi}
\end{aligned}
$$

Substituting the numbers computed above in the expression for y, we have for the equation of the trajectory,

$$y = 0.062383\{0.40097z - [A(u) - 213.15]\}$$

and in connection therewith,

$$S(u) = z + 5393.8$$

To compute the co-ordinates of the summit we have (since $I(u_0) = 0.40097$), $u_0 = 851.64$; and from Table I, $S(u_0) =$ 10038.3 and $A(u_0)$ 1297.81

$$\therefore\ z_0 = 1003.3 - 53983.8 = 4644.5$$

$$
\begin{aligned}
\therefore\ y_0 &= 0.062383\ (0.40097 \times 4644.5 - 1084.66) \\
&= 0.062383 \times 777.65 = 48.5 \text{ feet.}
\end{aligned}
$$

$$\therefore\ x_0 = Cz_0 = 1638.3 \text{ feet.}$$

The following table gives the value of y, and the velocity of the bullet, for each 100 yards, computed by the above formula:

x yards	Velocity in feet per sec.	y feet	x yards	Velocity in feet per sec.	y feet
0	1301.00	0.0	546.1	851.64	48.5
100	1161.25	14.8	600	827.39	47.9
200	1054.86	27.4	700	785.89	43.5
300	983.22	37.4	800	747.47	34.4
400	925.14	44.5	900	710.92	20.1
500	873.54	48.1	1000	676.17	0.0

Example 4. A 3.2-inch projectile, weighing 13 pounds, is fired with a muzzle velocity of 985 f. s. How high should a

target be at 80 feet from the gun in order that the projectile shall pass through it and strike a target on the same level as the gun and at a distance of 1200 yards?

Determine also the elevation necessary for the purpose.

Here $d = 3.2$, $w = 13$. $c = 0.9$, $\log C = 0.14940$, $V = 986$, $X = 3600$ and $x = 80$.

Required ϕ and y.

$$\begin{aligned} \log X &= 3.55630 \\ \log C &= 0.14940 \\ \hline \log z &= 3.40690 \end{aligned}$$

$$\begin{aligned} \therefore z &= 2552.1 \\ S(V) &= 7907.1 \\ \hline S(u) &= 10459.2 \qquad \therefore u = 829.33 \end{aligned}$$

$$\begin{aligned} A(u) &= 1474.55 \\ S(V) &= 626.96 \\ \hline \end{aligned}$$

$$\begin{aligned} \log 847.59 &= 2.92819 \\ \log z &= 3.40690 \\ \hline \log 0.33212 &= 9.52129 \qquad = \log I(u_0) \\ I(V) &= 0.23655 \\ \hline \log 0.09557 &= 8.98032 \\ \log C &= 0.14940 \\ \hline \log \sin 2\phi &= 9.12972 \qquad \therefore \phi = 3^\circ\ 50'\ 20'' \end{aligned}$$

$$\begin{aligned} \log x &= 1.90309 \\ \log C &= 0.14940 \\ \hline \log z &= 1.75369 \end{aligned}$$

$$\begin{aligned} z &= 56.7 \\ S(V) &= 7907.1 \\ \hline S(u) &= 7963.8 \qquad \therefore u = 981.87 \end{aligned}$$

$$\begin{aligned}
\log z &= 1.75369 \\
\log I(u_0) &= 9.52129 \\
\hline
\log 18.84 &= 1.27498 \\
A(V) &= 626.96 \\
\hline
& 645.80 \\
A(u) &= 640.49 \\
\hline
\log 5.31 &= 0.72509 \\
2 \log C &= 0.29880 \\
\log 0.5 &= 9.69897 \\
2 \log \sec. \phi &= 0.00195 \\
\hline
\log y &= 0.72481 \qquad \therefore y = 5.306 \text{ feet.}
\end{aligned}$$

Example 5. Compute ordinates for plotting the mean trajectory of the six shots fired from the 8-inch Pneumatic Torpedo gun, Sept. 20, 1887, at the schooner "Silliman."

The data furnished by Captain Zalinski, in charge of the firing, are as follows:

Diameter of projectile 7.75-inches; weight of projectile (including charge of 55 pounds of explosive-gelatine and dynamite), $137\frac{1}{6}$ pounds; mean range of the six shots, 5583.5 feet; angle of departure $14° 53' 20''$; time of flight 10.63 seconds.

From this data were computed, by a laborious, tentative process, the muzzle velocity and the ballistic coefficient as follows: $V = 765.7$ f. s. and $C = 9.70378$; whence it follows that the coefficient of reduction is 4.5172. In other words, this projectile, with its spiral wings for imparting rotation, and its very unsteady flight, suffers a resistance more than $4\frac{1}{2}$ times as great as a service projectile of the same diameter and moving with the same velocity. It was also found that the angle of fall was $22° 12'$, and striking velocity 416.9 f. s.

Proceeding as in the other examples we find the expressions for u and y to be

$$S(u) = z + 11789.0$$

and

$$y = 0.136833(1.55666z + 2146.03 - A(u)).$$

The following table gives the values of y for every 200 yards, and also the co-ordinates of the summit.

x in yards.	z	u	$A(u)$	y in feet.
0	0	......	....	0.00
200	1186.8	713.97	2908.58	148.45
400	2373.6	665.74	3847.90	272.71
600	3560.4	620.77	4994.16	368.66
800	4747.2	578.83	6375.00	432.50
1000	5934.0	539.72	8026.29	459.34
1030.1	6112.8	534.07	8301.46	459.78
1200	7120.8	503.26	9989.96	443.44
1400	8307.6	469.26	12311.5	378.56
1600	9494.3	437.57	15044.0	257.43
1800	10681.1	408.00	18252.4	71.21
1861.2	11044.1	399.37	19338.0	0.00

Approximate Expressions for y.—In the ascending branch of the trajectory, we may determine an approximate value of y when x is small, by the equation

$$y = x \tan \phi.$$

If, in the descending branch, we make

$$X - x = \Delta X,$$

we shall have approximately, when ΔX is small,

$$y = \Delta X \tan \omega.$$

PROBLEM XVI.

Given the elements of a trajectory and any ordinate (y), *to compute the corresponding abscissa* (x).

Solution. As our equations give no simple relation between y and u (such, for example, as that given by equation (1) between x and u), it is necessary to solve this problem by approximations. By combining equations (1), (3) and (6) we obtain the following equation:

$$I(u_0)S(u) - A(u) = \frac{2 \cos^2 \phi}{C^2} y + I(u_0)S(V) - A(V),$$

in which u_0 refers to the summit of the trajectory, and u to the point (x, y). $I(u_0)$ will be computed by the equation

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V);$$

and, therefore, the second member of the above equation contains only known quantities.

Having determined by trial the value (or values) of u which satisfy the above equation, we find x by the equation

$$x = C\{S(u) - S(V)\}.$$

Example 1. Compute the values of x for which $y = 200$ feet, with the data of Ex. 5, Prob. XV.

The computations are as follows:

$$\begin{aligned} \log \sin 2\phi &= 9.69604 \\ \log C &= 9.70378 \\ \hline \log 0.98235 &= 9.99226 \end{aligned}$$

$$\begin{aligned} I(V) &= 0.57423 \\ \hline (Iu_0) &= 1.55658 \end{aligned}$$

We also find

$$\frac{2\cos^2\phi}{C^2}y = 1461.6$$
$$I(u_0)S(V) = 18350.5$$
$$A(V) = 2146.0$$

We therefore have the equation

$$1.55658S(u) - A(u) = 17666.1$$

from which to find u by a method similar to that given on page 64.

By a few trials we find this equation is satisfied when $u = 694.73$ and $u = 426.98$,—the first value referring to the ascending and the second to the descending branch. The two values of x are now computed as follows:

$$\begin{aligned} S(694.73) &= 13439.3 \\ S(V) &= 11789.0 \end{aligned}$$

$$\begin{aligned} \log 1650.3 &= 3.21756 \\ \log C &= 9.70378 \\ \log x &= 2.92134 \qquad \therefore x = 834.3 \text{ feet.} \end{aligned}$$

$$\begin{aligned} S(426.98) &= 21699.0 \\ S(V) &= 11789.0 \end{aligned}$$

$$\begin{aligned} \log 9910.0 &= 3.99607 \\ \log C &= 9.70378 \\ \log x &= 3.69985 \qquad \therefore x = 5010.2 \text{ feet.} \end{aligned}$$

Practical Applications.—The only practical applications of this problem relate to that part of the trajectory near the striking point; such, for example, as calculating the breadth of the danger-zone, etc. We will, therefore, change the expres-

sion for x to one better suited to the purpose. We have

$$x = C\{S(u) - S(V)\},$$

and

$$X = C\{S(u_\omega) - S(V)\};$$

$$\therefore X - x = \varDelta X = C\{S(u_\omega) - S(u)\}.$$

Example 2. The small-calibre magazine carbine manufactured at the Austrian Factory at Steyer, in 1886, for the Portuguese Government (Kropatschek system), has the following characteristics: Calibre 0.315 inch; length of bullet 4 calibres; weight of bullet 246.9 grains; twist of rifling one turn in 11.024 inches; muzzle velocity 1608 f. s. Compute $\varDelta X$ when $y = 5.75$ feet: and also when $y = -5.75$ feet, *for a range of* 1000 *yds.*

Here we have $d = 0.315$, $w = \dfrac{246.9}{7000}$, $c = 0.9$, $\log C = 9.59656$, $X = 3000$ and $V = 1608$. We will first compute ϕ and ω by Problem XII.

$$\begin{aligned}
\log X &= 3.47712 \\
\log C &= \underline{9.59656} \\
\log z &= 3.88056 \\
z &= 7595.7 \\
S(V) &= \underline{3903.7} \\
S(u_\omega) &= 11499.4 \qquad \therefore u_\omega = 778.881
\end{aligned}$$

$$\begin{aligned}
A(u_\omega) &= 1984.28 \\
A(V) &= \underline{93.77}
\end{aligned}$$

$$\begin{aligned}
\log 1890.51 &= 3.27658 \\
\log z &= \underline{3.88056} \\
\log 0.24890 &\doteq 9.39602 = \log I(u_0) \\
I(V) &= \underline{0.05867} \\
&\ 0.19023
\end{aligned}$$

$$\begin{aligned}\log 0.19023 &= 9.27928\\ \log C &= 9.59656\\ \hline \log \sin 2\phi &= 8.87584 \qquad \therefore \phi = 2^\circ\ 9'\ 16''\end{aligned}$$

$$\begin{aligned}I(u_\omega) &= 0.54300\\ I(u_0) &= 0.24890\\ \hline \log 0.29410 &= 9.46850\\ \log C &= 9.59656\\ \hline \log \sin 2\omega &= 9.06506 \qquad \therefore \omega = 3^\circ\ 20'\ 7''\end{aligned}$$

Substituting numbers already found in the equation

$$I(u_0)S(u) - A(u) = \frac{2\cos^2\phi}{C^2}y + I(u_0)S(V) - A(V);$$

it becomes when $y = 5.75$,

$$I(u_0)S(u) - A(u) = 73.62 + 971.63 - 93.77;$$

or

$$0.24890S(u) - A(u) = 951.48,$$

from which to find u.

By a few trials we find that this equation is satisfied when $u = 1519.64$ and $u = 791.03$. The first of these values refers to the ascending branch, and is of no practical importance in this example. Using the second value we compute ΔX by the equation

$$\Delta X = C\{S(u_\omega) - S(u)\}$$

as follows:

$$\begin{aligned}S(u_\omega) &= 11499.4\\ S(u) &= 11236.9\\ \hline \log 262.5 &= 2.41913\\ \log C &= 9.59656\\ \hline \log \Delta X &= 2.01569 \qquad \therefore \Delta X = 103.7 \text{ feet.}\end{aligned}$$

Next let $y = -5.75$ feet. In this case we have

$$I(u_0)S(u) - A(u) = -73.62 + 971.63 - 93.77;$$

or

$$0.24890S(u) - A(u) = 804.24,$$

from which we easily find $u = 1696.7$ and $u = 767.94$, as the two values of u which satisfy this equation. The first value of u belongs to a point in the ascending branch prolonged backward through the origin; and the second value to a point in the descending branch prolonged through the 1000-yard point, or point of fall.

Using this second value of u we find ΔX as follows:

$$\begin{array}{rl} S(u) = & 11739.3 \\ S(u_\omega) = & \underline{11499.4} \\ \log 239.9 = & 2.38003 \\ \log C = & \underline{9.59656} \\ \log \Delta X = & 1.97659 \qquad \therefore \Delta X = 94.75 \text{ feet.} \end{array}$$

The first value of ΔX (103.7 feet) is the breadth of the danger-zone on level ground, against infantry, when the gun is fired with its muzzle close to the ground, and aimed at the foot of the target. This zone therefore lies entirely *within* the 1000-yard range. The second value of ΔX (94.75 feet) is the breadth of the danger-zone when the gun is fired with its muzzle at a height of 5.75 feet above the ground and at a point of the target at the same height. This zone therefore lies entirely *without* the 1000-yard range.

The actual danger-zone lies partly within and partly without the range point; and its breadth is a certain mean of the two, computed as above, depending upon the height of the muzzle of the gun.

Application of the Principle of the Rigidity of the Trajectory.—The essential features of the principle of the *rigidity of the trajectory* may be concisely stated as follows:—(see page 101.)

If, for a certain gun, ϕ', ω' and u'_ω refer to a given *horizontal* range (or chord) s, then the corresponding elements of a trajectory which shall pass through a point at the same distance s, but which is above (or below) the gun by the angular distance ϵ, may be determined by the relations

$$\phi = \phi' + \epsilon,$$

$$\theta = -\omega' + \epsilon,$$

$$u_\theta = u'_\omega.$$

According to this principle a rifle should be sighted (within the prescribed limits) for *distance* only; that is, without reference to the angular elevation (or depression) of the object above (or below) the level of the gun; and then aimed directly at the object; for, it is evident, that to the elevation, denoted by ϕ' (to which the sights are set), ϵ is added by simply pointing the gun at the object.

In laying heavy guns with the Zalinski sight, the vernier should be set to ϕ' (taken from the Table of Fire, for the given horizontal distance), and the gun then so manœuvred that the axis of the telescope is directed on the object; when if the jump of the piece has also been taken into account, the gun will have the proper elevation.

Example 3. What would be the danger-space against Infantry ($h = 5.75$ feet), on level ground, in the preceding example, if the muzzle of the gun were 2 feet high (as in firing kneeling), and aimed at a point 4 feet from the ground and 1000 yards distant?

We have already found (page 122)

$$\phi' = 2^\circ\ 9'\ 16'' \qquad \text{(Sighting angle)}$$

$$\omega' = 3^\circ\ 20'\ 7''$$

$$u'_\omega = 778.88 \text{ f. s.}$$

We also have by a given condition,

$$\tan \epsilon = \frac{2}{3000};$$

whence $\epsilon = 2' \ 18''$.

Therefore, for the new trajectory we have

$$\phi = 2^\circ \ 11' \ 34''$$
$$\theta = -\ 3^\circ \ 17' \ 49''$$
$$u_\theta = 778.88 \text{ f. s.}$$

We have next to compute values of u in the descending branch for which $y = 3.75$ feet and $y = -2$ feet. If u' and u'' are these values, we shall have for the danger-space ΔX,

$$\Delta X = C\{S(u'') - S(u')\}.$$

The equations for determining u' and u'' by trial are found to be

$$0.25228S(u') - A(u') = 939.07,$$

and

$$0.25228S(u'') - A(u'') = 865.46;$$

from which we find

$$u' = 782.46$$

and

$$u'' = 770.94$$

$$S(u'') = 11673.2$$
$$S(u') = 11421.5$$

$$\log 251.7 = 2.40088$$
$$\log C = 9.59656$$

$$\log \Delta X = 1.99744 \qquad \therefore \Delta X = 99.4 \text{ feet.}$$

Example 4. Compare the *maximum* danger-spaces against infantry covered by the Steyer carbine and Springfield rifle, respectively, for angles of elevation corresponding to different ranges up to 1000 yards, reckoning from the muzzle of the gun.

The results of the calculations, with the data upon which they are based, are given in the tables below. We have taken

STEYER CARBINE.

Data: $V = 1608$ f. s.; $w = 246.9$ grains; $d = 0.315$ inches; $c = 1$; $\log C = 9.55080$.

Range, in yards.	Angle of departure.	Angle of fall.	Final velocity, in feet per second.	Maximum ordinate, in feet.	Danger-space in ascending branch, in feet.	Danger-space in descending branch, in feet.	Total maximum danger-space, in yards.
	° ′ ″	° ′ ″					
100	0 6 57	0 7 33	1426.30	0.16	153	847	333
200	0 15 9	0 17 46	1265.57	0.74	312	764	355
300	0 24 48	0 31 20	1133.86	1.83	477	722	400
400	0 36 5	0 48 27	1037.06	3.67	645	684	443
500	0 49 00	1 8 35	970.45	6.43	568	453	340
600	1 3 29	1 31 32	914.25	10.22	359	263	207
700	1 19 32	1 57 32	864.19	15.20	269	193	154
800	1 37 8	2 26 52	819.32	21.53	214	146	120
900	1 56 18	2 59 44	778.89	29.37	176	117	98
1000	2 17 10	3 36 31	741.09	38.93	148	95	81

SPRINGFIELD RIFLE.

Data: $V = 1301$ f. s.; $w = 500$ grains; $d = 0.45$ inches; $c = 1$; $\log C = 9.54745$.

Range, in yards.	Angle of departure.	Angle of fall.	Final velocity, in feet per second.	Maximum ordinate, in feet.	Danger-space in ascending branch, in feet.	Danger-space in descending branch, in feet.	Total maximum danger-space, in yards.
	° ′ ″	° ′ ″					
100	0 10 36	0 11 26	1161.25	0.24	153	676	276
200	0 22 55	0 26 23	1054.86	1.07	311	645	319
300	0 36 56	0 44 28	983.22	2.67	471	619	363
400	0 52 32	1 5 24	925.14	5.18	632	597	410
500	1 9 39	1 29 23	873.54	8.74	345	284	210
600	1 28 19	1 56 41	827.39	13.51	249	195	148
700	1 48 36	2 27 30	785.89	19.67	193	147	113
800	2 10 34	3 2 7	747.47	27.41	158	115	91
900	2 34 17	3 40 59	710.92	36.95	132	94	75
1000	2 59 53	4 24 35	676.17	48.51	109	77	62

$c = 1$, in the absence of any accurate data for determining its true value. It is probably less than unity for the new small-calibre rifles.

In computing the maximum danger-spaces for the first four ranges in the above tables, the gun was supposed to be raised so as to bring the summits of the trajectories 5.75 feet from the ground.

For example, with the Steyer carbine and a range of 100 yards, the maximum ordinate is 0.16 feet. In this case the muzzle of the gun must be raised $5.75 - 0.16 = 5.59$ feet from the ground to obtain the maximum danger-space with an angle of departure due to a range of 100 yards. In like manner for ranges of 200, 300 and 400 yards, the muzzle must be at the heights, respectively, of 5.01, 3.92 and 2.08 feet to give the greatest danger-spaces.

For a range of 500 yards the height of the summit is 8.74 feet; and for this and greater ranges the muzzle of the gun must be on a level with the ground to produce the maximum danger-space.

Example 5. A skirmisher, firing kneeling, with a Springfield rifle, fixes his sight for a target 500 yards distant, and aims at a point of the target at the same height from the ground as the muzzle of his gun, which is 2.5 feet. What is the breadth of the danger-space against infantry covered by his bullet?

Here we have from Ex. 4, for a horizontal range of 500 yards, $V = 1301$ f. s.; $u_\omega = 873.54$; $\phi = 1° \; 9' \; 39''$; and $\log C = 9.54745$. With this data we find by (6)

$$I(u_0) = 0.21968.$$

If u' refer to the point of the trajectory where $y' = 3.25$ feet $(= 5.75 - 2.5)$; and u'' to the point where $y = -2.5$ feet, we have, for computing the danger-zone, the equation

$$\Delta X = C\{S(u'') - S(u')\}.$$

Substituting known quantities in the equation

$$I(u_0)S(u) - A(u) = \frac{2\cos^2\phi}{C^2}y + I(u_0)S(V) - A(V),$$

we have, for computing u' and u'', the equations

$$0.21968\ S(u') - A(u') = 1023.98$$

and

$$0.21968\ S(u'') - A(u'') = 931.59,$$

from which we find

$$u' = 897.01,$$
$$u'' = 859.22;$$

and then

$$\Delta X = 230.4 \text{ feet} = 76.8 \text{ yards.}$$

To determine where the danger-space begins and ends with reference to the point of fire, we evidently have the equations

$$x' = C\{S(u') - S(V)\}$$

and

$$x'' = C\{S(u'') - S(V)\},$$

from which are deduced

$$x' = 453.1 \text{ yards},$$
$$x'' = 529.9 \text{ yards};$$

that is to say, the danger-space begins 46.9 yards in front of the target, and ends 29.9 yards beyond it.

Example 6. Suppose the skirmisher in the preceding example aims at a point of the target 4 feet from the ground: what is the danger-space?

To determine the angle of departure in this case we must add to the former value of ϕ ($1°\ 9'\ 39''$) the angle of elevation of the point aimed at above the horizontal plane passing through the muzzle of the gun (ϵ). We have

$$\tan \epsilon = \frac{4 - 2.5}{3 \times 500} = \frac{1.5}{1500}.$$

$$\therefore\ \epsilon = 3'\ 26'';$$

$$\therefore\ \phi = 1°\ 13'\ 05''.$$

Whence, by (6),

$$I(u_0) = 0.22532.$$

The equations for computing u' and u'' are next found to be

$$0.22532\, S(u') - A(u') = 1054.40,$$
$$0.22532\, S(u'') - A(u'') = 962.01,$$

from which we obtain

$$u' = 885.88,$$
$$u'' = 851.02,$$

and then

$$\Delta X = 217.25 \text{ feet} = 72.42 \text{ yards}.$$

We also find for the beginning and ending of the danger-space,

$$x' = 475.0 \text{ yards},$$
$$x'' = 547.4 \text{ yards}.$$

Approximate Method for Computing the Danger-space.—The danger-space is usually computed by the formula

$$\Delta X = \frac{y}{\tan \omega},$$

which gives but a rough approximation when applied to the flat trajectories of the modern rifle. For example, the danger-space in the descending branch, for a range of 500 yards, with the Steyer carbine, is 453 feet; while the danger-space computed by the above formula is but 288 feet—an error of 165 feet, or 36 per cent. The error, however, rapidly diminishes as the angle of fall increases, and also as the value of C increases. This method is therefore better adapted to sea-coast than to small-arm firing.

Example 7. Using the 8-inch B. L. rifle, compute the danger-space for the side of a ship which projects 20 feet out of the water, at a range of 2 miles.

Here we have $V = 1850$ f. s.; $X = 10{,}560$ feet; and $\log C = 0.70198$. (See page 19.)

If we have a Table of Fire, we should find the danger-space by dividing 20 by the tangent (or multiplying by the cotangent)

of the angle of fall, taking this latter directly from the Table. Without such a Table the calculations will be as follows:

$$\begin{aligned}
\log X &= 4.02366 \\
\log C &= 0.70198 \\
\hline
\log z &= 3.32168 \\
\therefore z &= 2097.4 \\
S(V) &= 2916.9 \\
\hline
S(u) &= 5014.3 \qquad \therefore u = 1373.28
\end{aligned}$$

$$\begin{aligned}
A(u) &= 176.01 \\
A(V) &= 46.93 \\
\hline
\log 129.08 &= 2.11086 \\
\log z &= 3.32168 \\
\hline
\log 0.06154 &= 8.78918 = \log I(u_0) \\
I(V) &= 0.03727 \\
\hline
\log 0.02427 &= 8.38507 \\
\log C &= 0.70198 \\
\hline
\log \sin 2\phi &= 9.08705 \qquad \therefore \phi = 3^\circ\ 30'\ 34'' \\
I(u) &= 0.09116 \qquad \text{(Equation 16)} \\
I(u_0) &= 0.06154 \\
\hline
2)0&.02962 \\
\hline
\log 0.01481 &= 8.17056 \\
\log C &= 0.70198 \\
\log \sec^2 \phi &= 0.00163 \\
\hline
\log \tan \omega &= 8.87417 \qquad \therefore \omega = 4^\circ\ 16'\ 50'' \\
\log 20 &= 1.30103 \\
\hline
\log \Delta X &= 2.42686
\end{aligned}$$

$$\therefore \Delta X = 267.2 \text{ feet} = 89.1 \text{ yards.}$$

The error in this example is less than one foot.

Sladen's Method of Computing Danger-spaces.—The following method is sometimes employed for computing the danger-space for flat trajectories, especially by English artillerists. It is known as *Sladen's Method*, having been first employed by Lt. Col. Sladen of the Royal Artillery.

We have *in vacuo* the following relation:

$$y = \frac{gt}{2}\left(T - t\right),$$

in which T is the time of flight for a given horizontal range, and t the time from the origin to a point of the trajectory whose ordinate is y. (See Appendix 1.)

Solving this equation with reference to t, we have

$$t = \tfrac{1}{2}\left\{T + \sqrt{T^2 - \frac{8y}{g}}\right\}.$$

This equation would give the exact values of t for given values of y and T, were there no resistance. It also gives a close approximation to the value of t for flat trajectories when, instead of the time of flight *in vacuo*, we use the *actual time of flight*, computed by the formula

$$T = \frac{C}{\cos\phi}\left\{T(u_\omega) - T(V)\right\}.$$

Having computed the value of t as above, we next find the corresponding value of u by the equation

$$T(u) = \frac{t}{C} + T(V),$$

and then ΔX by the equation

$$\Delta X = C\{S(u_\omega) - S(u)\}.$$

Example 8. Compute the danger-space against infantry for the Springfield rifle, at a range of 500 yards, the muzzle of the gun being on a level with the ground, by Sladen's Method.

We have from Ex. 4, $V = 1301$ f. s. and $u_\omega = 873.54$; also $y = 5.75$ feet and $g = 32.16$.

$$
\begin{aligned}
T(u_\omega) &= 7.035 \\
T(V) &= \underline{2.896} \\
\log 4.139 &= 0.61690 \\
\log C &= \underline{9.54745} \\
\log T &= 0.16435 \qquad \therefore\ T = 1.460 \text{ sec.}
\end{aligned}
$$

$$
\begin{aligned}
T^2 &= 2.1316 \\
\frac{8y}{g} &= \underline{1.4303} \\
\log 0.7013 &= 9.84590 \qquad \text{(Divide by 2)} \\
\log 0.8374 &= 9.92295 \\
T &= \underline{1.4600} \\
&\ 2)2.2974 \\
t &= 1.1487 \\
\log t &= 0.06021 \\
\log C &= \underline{9.54745} \\
\log 3.257 &= 0.51276 \\
T(V) &= \underline{2.896} \\
T(u) &= 6.153 \qquad \therefore\ u = 921.30
\end{aligned}
$$

$$
\begin{aligned}
S(u_\omega) &= 9646.4 \\
S(u') &= \underline{8856.0} \\
\log 790.4 &= 2.89785 \\
\log C &= \underline{9.54745} \\
\log \Delta X &= 2.44530 \\
\Delta X &= 279 \text{ feet} = 93 \text{ yards.}
\end{aligned}
$$

This differs but 5 feet from the danger-space computed by the more rigorous method first employed. There is not much difference in the labor required by the two methods.

Danger-range.—When the danger-space coincides with the range, it is called the *danger-range.* This is an important unit in Gunnery for comparing the efficiency of different guns. To determine the length of the danger-range we have to compute the horizontal trajectory whose maximum ordinate (y_0) is given. As the angle of departure (ϕ) is not known in this case, we cannot compute $I(u_0)$ as heretofore, by means of the equation

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V);$$

we will, therefore, change the equation

$$I(u_0)\, S(u) - A(u) = \frac{2\cos^2\phi}{C^2} y + I(u_0)\, S(V) - A(V)$$

to the following equivalent form, which relates to the summit:

$$A(u_0) - I(u_0)\{S(u_0) - S(V)\} = A(V) - \frac{2\cos^2\phi}{C^2} y_0.$$

The second member of this equation consists of known quantities, with the exception of the fact or $\cos^2\phi$. But, as ϕ is very small in all danger-ranges, rarely exceeding $1°$, we may take $\cos^2\phi$ as unity without impairing the accuracy of the results. We have, then, first to compare the value of u_0 by trial, from the equation

$$A(u_0) - I(u_0(\{S(u_0) - S(v)\} = A(V) - \frac{2y_0}{C^2};$$

and then all the other elements of the trajectory may be determined by methods already given.

Example 9. Compute the danger-ranges against infantry, for the Steyer carbine and Springfield rifle.

We have for the Springfield rifle, using data already given,

$$A(V) = 213.15$$

$$\frac{2y_0}{C^2} = 92.43$$

$$\text{Difference} = 120\ 72$$

We therefore have the equation

$$A(u_0) - I(u_0)\{S(u_0) - 5393.8\} = 120.72$$

from which to find u_0 by trial. It will be seen, by referring to the table of Ex. 4, that the required range lies between 400 and 500 yards, and considerably nearer the former than the latter. For a range of 400 yards we have $u_0 = 1046$; and for a range of 500 yards, $u_0 = 1006$. We will therefore assume $u_0 = 1046$ for a first trial. We then proceed as follows:

$$\begin{array}{rl}
S(1040) = & 7247.9 \\
S(1301) = & \underline{5393.8} \\
\log 1854.1 = & 3.26813 \\
\log I(1040) = & \underline{9.29028} \\
\log 361.75 = & 2.55841 \\
A(1040) = & \underline{484.95} \\
& 123.20 \\
\text{True value} & \underline{120.72} \\
\text{Error} = & -2.48
\end{array}$$

As the numerical value of the first member of the above equation decreases with the argument, we will next assume $u_0 = 1037$.

$$\begin{array}{rl}
S(1037) = & 7280.1 \\
S(1301) = & \underline{5393.8} \\
\log 1886.3 = & 3.27561 \\
\log I(1037) = & \underline{9.29453} \\
\log 371.66 = & 2.57014 \\
A(1037) = & \underline{491.28} \\
& 119.62 \\
\text{True value} & \underline{120.72} \\
\text{Error} = & +1.10
\end{array}$$

Therefore, by the rule on page 64,

$$358 : 3 :: 110 : 0.92.$$
$$\therefore u_0 = 1037.92,$$

which completely satisfies the above equation.

To compute the angle of departure we have, from (6),

$$\sin 2\phi = C\{I(u_0) - I(V)\}.$$

$$\begin{aligned} I(u_0) &= 0.19644 \\ I(V) &= 0.10483 \\ \hline \log 0.09161 &= 8.96194 \\ \log C &= 9.54745 \\ \hline \log \sin 2\phi &= 8.50939 \qquad \therefore \phi = 0° \, 55' \, 33'' \end{aligned}$$

To determine the striking velocity we make use of (14), which becomes, by applying numbers already calculated,

$$\frac{A(u_\omega) - 213.15}{S(u_\omega) - 5393.8} = 0.19644.$$

From this equation we find by trial, as explained in the second method of Problem IX,

$$u_\omega = 915.24.$$

The danger-range is finally computed as follows:

$$\begin{aligned} S(u_\omega) &= 8950.6 \\ S(V) &= 5393.8 \\ \hline \log 3556.8 &= 3.55106 \\ \log C &= 9.54745 \\ \hline \log X &= 3.09851 \end{aligned}$$

$\therefore X = 1255$ feet $= 418.3$ yards, which is the danger-range for the Springfield rifle.

For the Steyer carbine we find, by similar calculations,

$$\phi = 0^\circ\ 46'\ 6'';$$
$$X = 1436 \text{ feet} = 478.7 \text{ yards}.$$

Example 10. Calculate the danger-range against cavalry, for the 3.2-inch B. L. rifle (steel).

Here $V = 1608$ f. s.; $y_0 = 8.25$ feet; $w = 13$ pounds; $d = 3.2$ inches; $c = 0.93$; $\log C = 0.13516$.

Proceeding as in Ex. 9, we deduce the equation

$$A(u_0) - I(u_0)\{S(u_0) - 3903.7\} = 84.92,$$

from which we find by trial

$$u_0 = 1438.77.$$

$$\begin{aligned} I(u_0) &= 0.08048 \\ I(V) &= \underline{0.05867} \end{aligned}$$

$$\begin{aligned} \log 0.02181 &= 8.33866 \\ \log C &= \underline{0.13516} \\ \log \sin 2\phi &= 8.47382 \quad \therefore \phi = 0^\circ\ 51'\ 11'' \end{aligned}$$

To determine the striking velocity we must solve by trial the equation

$$\frac{A(u_\omega) - 93.77}{S(u_\omega) - 3903.7} = 0.08048;$$

from which we find

$$u_\omega = 1297.00.$$

$$\begin{aligned} S(u_\omega) &= 5415.6 \\ S(V) &= \underline{3903.7} \end{aligned}$$

$$\begin{aligned} \log 1511.9 &= 3.17952 \\ \log C &= \underline{0.13516} \\ \log X &= 3.31468 \quad \therefore X = 2064 \text{ feet} = 688 \text{ yards}. \end{aligned}$$

Example 11. Calculate the danger-range for the 8-inch B. L. rifle, supposing the target to be a ship's side projecting 12 feet out of the water.

We have $V = 1850$ f. s., $\log C = 0.70198$ and $y_0 = 12$ feet.

To determine u_0 we have the equation

$$A(u_0) - I(u_0)\{S(u_0) - 2916.9\} = 45.98,$$

from which we find

$$u_0 = 1770.4;$$

and for the final velocity

$$\frac{A(u_\omega) - 46.93}{S(u_\omega) - 2916.9} = 0.04335,$$

whence

$$u_\omega = 1696.1.$$

We also find

$$\phi = 0^\circ\ 52'\ 35'';$$
$$X = 1026.2 \text{ yards.}$$

Point-blank Firing.—If we suppose the axis of a gun horizontal when fired, and at a distance of y feet above the ground, we can determine the point where the projectile will strike the ground (also considered horizontal), as follows:

We have, in this case, $u_0 = V$ and $\phi = 0$; whence our general equation becomes

$$I(V)S(u) - A(u) = -\frac{2y}{C^2} + I(V)S(V) - A(V),$$

from which to determine u by trial.

Example 12. The Springfield rifle is fired parallel with the ground (no elevation) and at a height of four feet above it. How far from the gun will the bullet strike the ground?

Here we have $y = 4$, and substituting numbers already given in the above equation, we have

$$0.10483\ S(u) - A(u) = 287.98;$$

from which we find by trial, as already explained,

$$u = 1054.85.$$

We now compute the range as follows:

$$\begin{aligned} S(u) &= 7094.9 \\ S(V) &= 5393.8 \\ \hline \log 1701.1 &= 3.23073 \\ \log C &= 9.54745 \\ \hline \log X &= 2.77818 \end{aligned}$$

$$\therefore\ X = 600 \text{ feet} = 200 \text{ yards.}$$

For the Steyer carbine we have $u = 1207.26$, and $X = 242$ yards.

By the principle of the Rigidity of the Trajectory the values of X in the above example would be the same also for ground not level, the other conditions remaining the same.

We may obtain an easier *approximate* solution of Ex. 12 as follows: The bullet must fall 4 feet (y) before it strikes the ground; requiring a time, t, determined by the equation

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{8}{2.16}} = 0.49876 \text{ seconds.}$$

Therefore by (2), (calling $\cos\phi$ unity,)

$$T(u) = \frac{t}{C} + T(V).$$

This gives for the Springfield rifle

$$u = 1060.33, \text{ and } X = 582 \text{ feet};$$

and for the Steyer carbine

$$u = 1220.4, \quad \text{and } X = 696 \text{ feet.}$$

These ranges differ, respectively, 18 and 30 feet from the more accurate ones previously computed.

Estimating Distances.—We will illustrate this by a few examples.

Example 13. A skirmisher armed with a Springfield rifle estimates his distance from the target at 500 yards, and sets his sights accordingly.

If the target is 5.75 feet high and he aims at a point at the same height as the muzzle of his gun, which is 2.5 feet, what error can he make in estimating the distance and still hit the target?

By comparing this with Ex. 5, it will be apparent that the target may be anywhere within the danger-space, which is there worked out. His estimate of the distance may therefore be too small by

$$500 - 453 = 47 \text{ yards,}$$

or too great by

$$530 - 500 = 30 \text{ yards,}$$

and still hit the target.

Example 14. "Fire is being carried out against the side of an iron-clad which projects 10 feet above the water-line, by a 12.5-inch gun of 38 tons, with a muzzle velocity of 1575 f. s., and a 809 lb. 6 oz. shot, from a battery 80 feet above the sea level and 1500 yards [estimated] from the ship. What error made in estimating the range will admit of the projectile striking the side?"

Here $V = 1575$, $d = 12.5$, $w = 809.375$, $s = 4500$, $c = 1$, and $\log C = 0.71433$.

This example admits of two cases: If there be a suspicion that the distance to the ship has been overestimated—that is, if the calculated trajectory may possibly be too high—the elevation will be given with reference to the water-line or foot of the target. On the other hand, if the gunner suspects that the distance may be more than 1500 yards he will aim at the top of the target.

First consider that the point aimed at is on the water-line

of the ship. Then we have to determine the actual trajectory described, from the data $s = 4500$ and $y = -80$, as follows:

$$\begin{aligned}
\log s &= 3.65321 \\
\log C &= 0.71433 \\
\hline
\log z &= 2.93888 \\
z &= 868.7 \\
S(V) &= 4049.9 \\
\hline
S(u_\omega) &= 4018.3 \qquad \therefore u_\omega = 1392.13
\end{aligned}$$

$$\begin{aligned}
A(u_\omega) &= 167.42 \\
A(V) &= 102.60 \\
\hline
\log 64.82 &= 1.81171 \\
\log z &= 2.93888 \\
\hline
\log 0.07462 &= 8.87283 = \log I(u_0) \\
I(V) &= 0.06238 \\
\hline
\log 0.01224 &= 8.08778 \\
\log C &= 0.71433 \\
\hline
\log \sin 2\phi' &= 8.80211 \qquad \therefore \phi' = 1^\circ\ 49'\ 3''
\end{aligned}$$

$$\begin{aligned}
I(u_\omega) &= 0.08792 \\
I(u_0) &= 0.07462 \\
\hline
\log 0.01330 &= 8.12385 \\
\log \frac{C}{2} &= 0.44330 \\
\log \sec^2 \phi &= 0.00044 \\
\hline
\log \tan \omega' &= 8.52759 \qquad \therefore \omega' = 1^\circ\ 58'\ 30'' \\
\log y &= 1.90309 \\
\log S &= 3.65321 \\
\hline
\log \sin \epsilon &= 8.24988 \\
\therefore \epsilon &= -1^\circ\ 1'\ 7'' \\
\therefore \theta &= -2^\circ\ 59'\ 37''
\end{aligned}$$

$$\begin{aligned} \log 10 &= 1.00000 \\ \log \tan \theta &= 8.71847 \\ \hline \log \Delta X &= 2.28153 \quad \therefore \Delta X = 191.2 \text{ feet.} \end{aligned}$$

The real distance of the ship may therefore be about 64 yards less than the estimated distance and still be hit. A similar calculation will show that it may also be about the same distance in excess, provided the angle of departure be calculated for the top of the target.

Example 15. In firing with a Springfield rifle on level ground, at a target 5.75 feet high, aim was taken at the middle point of the target, and the muzzle of the gun was held at the same height from the ground, viz., 2.875 feet. Suppose the distance to the target (unknown to the marksman) to be exactly 500 yards. What are the limits of error in his estimate of the distance within which the target may be hit?

It is evident that all the trajectories lying between those touching the top and bottom of the target, respectively, will pass through the target, while all those outside these limits will not touch the target. The limiting angles of departure are therefore

$$\phi' + \epsilon$$

and

$$\phi' - \epsilon,$$

in which ϕ' is the angle of departure for a horizontal range of 500 yards, and ϵ the vertical angle subtended at the point of firing by one half the target. We have then to compute the horizontal range due to each of these angles of departure and see how much they differ from 500 yards. From the table on page 126 we find, for a range of 500 yards,

$$\phi = 1^\circ\ 9'\ 39'';$$

we also have

$$\tan \epsilon = \frac{2.875}{1500}.$$

$$\therefore \epsilon = 0^\circ \ \ 6' \ 35''.$$

$$\text{and } \phi' + \epsilon = 1^\circ \ 16' \ 14''$$

$$\text{Therefore } \phi' - \epsilon = 1^\circ \ \ 3' \ \ 4''$$

are the limiting angles of departure.

The horizontal ranges due to these angles of departure may be computed by the method given on page 61. But they are more easily determined by interpolation from the Table of Ex. 4, using only first differences. In this way we find these ranges to be 535 yards and 462 yards, respectively. The marksman may therefore overestimate the distance to the target up to 35 yards, and underestimate it to 38 yards, and not miss. In this solution it is assumed that the marksman sets his sights accurately to correspond to the estimated distance, and "holds" directly upon the centre of the target.

The following Table gives the results of similar calculations for the Springfield rifle and Steyer carbine for ranges extending from 100 yards to 1000 yards. It is a continuation of the Tables on page 126.

	Steyer Carbine.		Springfield Rifle.	
Range.	Range can be overestimated (yards).	Range can be underestimated (yards).	Range can be overestimated (yards).	Range can be underestimated (yards).
100	330	Point Blank	243	Point Blank
200	160	" "	116	" "
300	97	116	71	77
400	64	73	49	53
500	45	51	35	38
600	34	38	27	29
700	27	29	21	23
800	21	23	17	19
900	18	19	14	15
1000	15	16	12	13

PROBLEM XVII.

Given the range (X), the final velocity (v) and the maximum ordinate (y_0), to compute the initial velocity (V) and the ballistic coefficient (C), for small angles of departure.

SOLUTION.

This problem can be solved only by successive approximations, as follows: Assume a value for V, with which and the given data compute y_0. If this computed value is too small, the assumed value of V is too great, and *vice versa*. Two, or at most four, trials will give the value of V with all desired accuracy, making use of the proportion given on page 64. The formulæ required are the following, given in the order in which they are to be used:

$$z = S(u) - S(V);$$

$$C = \frac{X}{z};$$

$$I(u_0) = \frac{A(u) - A(V)}{z};$$

$$\sin 2\phi = C\{I(u_0) - I(V)\};$$

$$z_0 = S(u_0) - S(V);$$

$$2y_0 = (C \sec \phi)^2\{z_0 I(u_0 + A(V) - A(u_0)\}.$$

Example 1. "A committee was recently appointed to recommend a new rifle which should give a velocity of 800 f. s. at 1000 yards range, with a maximum height of the trajectory of 32 feet. Determine the proportion of weight of bullet to calibre to fulfil these conditions."*

Here $v = u = 800$, $X = 3000$ and $y_0 = 32$, to find V and C.

Assume the muzzle velocity to be 1890 f. s.; then, by a calculation similar to that worked out below, we find the height of the summit to be 31.661 feet; or an error of $32 - 31.661 = 0.339$. Next assume $V = 1830$; and we find $y_0 = 32.284$ feet, which is in error by -0.284.

---

* Proceedings Royal Artillery Institution, No. 14, Vol. XV.

The true muzzle velocity is, therefore, found to be 1857 f. s. With this and the other data as given, viz., $v = u = 800$, and $X = 3000$, we work out the value of C and height of summit, in the following manner:

$$S(u) = 11048.0$$
$$S(V) = 2890.3$$

$$\log 8157.7 = 3.91157 = \log z$$
$$\log X = 3.47712$$

$$\log C = 9.56555$$
$$A(u) = 1749.84$$
$$A(V) = 45.95$$

$$\log 1703.89 = 3.23144$$
$$\log z = 3.91157$$

$$\log 0.20887 = 9.31987 = \log I(u_0) \quad \therefore u_0 = 1019.78$$
$$I(V) = 0.03677$$

$$\log 0.17210 = 9.23578$$
$$\log C = 9.56555$$

$$\log \sin 2\phi = 8.80133 \qquad \therefore \phi = 1^\circ\ 48'\ 52''$$
$$S(u_0) = 7474.7$$
$$S(V) = 2890.3$$

$$\log 4584.4 = 3.66128 = \log z_0$$
$$\log I(u_0) = 9.31987$$

$$\log 957.52 = 2.98115$$
$$A(V) = 45.95$$

$$1003.47$$
$$A(u_0) = 530.74$$

$$\log 472.73 = 2.67461$$
$$2 \log (C \sec \phi) = 9.13154$$

$$\log (2y_0) = 1.80615 \qquad \therefore y_0 = 31.998$$

This value of y_0 is practically 32 feet; so that we have, to satisfy the given conditions,

$$V = 1887 \text{ f. s.};$$
$$\log C = 9.56555;$$
$$\phi = 1^\circ\ 48'\ 52''.$$

Relation between Weight and Calibre of Bullet.—To determine the relation between the weight of bullet and calibre we have, omitting the factors $\frac{\delta_1}{\delta}$ and c,

$$C = \frac{w}{d^2},$$

in which w is in pounds and d in inches. If w is in grains, we have

$$C = \frac{w}{7000d^2};$$

whence

$$w = 7000Cd^2,$$

and

$$d = \left(\frac{w}{7000C}\right)^{\frac{1}{2}}.$$

In these equations C is given; and we can therefore compute the value of w for any assumed value of d, or the value of d for any assumed value of w.

Suppose the calibre to be that of the Steyer carbine, viz., 0.315 inch ($= d$): what must be the weight of projectile to fulfil the required conditions?

We have

$$\begin{aligned} \log 7000 &= 3.84510 \\ \log C &= 9.56555 \\ \log d^2 &= \underline{8.99662} \\ \log w &= 2.40727 \end{aligned}$$

$$\therefore\ w = 255.4 \text{ grains.}$$

This is but 8.5 grains heavier than the Steyer carbine projectile. So that this weapon would fulfil the imposed conditions by increasing the muzzle velocity from 1608 f. s. to 1857 f. s., and adding 8.5 grains to the weight of the projectile. This weapon, therefore, is not as powerful as that contemplated by the Committee.

The rifles giving the flattest trajectories at the present time are the Hebler (two-part cartridge) and the Lebel. For these rifles we have the following data:

HEBLER.	LEBEL.
$d = 0.296$ inches;	$d = 0.314$ inches;
$w = 225.31$ grains;	$w = 231.48$ grains;
$V = 1968.54$ f. s.	$V = 2200$ f. s.
For a range of 1000 yards,	For a range of 1000 yards,
$\phi = 1^\circ\ 39'\ 23''$;	$\phi = 1^\circ\ 29'\ 08''$;
$v_\omega = 819.8$ f. s.;	$\omega = 2^\circ\ 44'$;
$y_0 = 29.642$ feet.	$v_\omega = 820$ f. s.;
	$y_0 = 27.6$ feet.

Both these rifles give flatter trajectories and greater striking velocities at 1000 yards than the rifle specified by the Committee, but with greater muzzle velocities.

The Hebler rifle would exactly fulfil the conditions imposed by reducing its muzzle velocity to 1857 f. s., as the following calculation shows: We have $d = 0.296$ inches, to calculate w from the given value of C.

$$\begin{aligned} \log 7000C &= 3.41065 \\ \log d^2 &= 8.94258 \\ \hline \log w &= 2.35323 \quad \therefore\ w = 225.44 \text{ gr.,} \end{aligned}$$

agreeing with the weight of the Hebler bullet.

Inverse Problem.—It would seem as if the muzzle velocity of the Lebel bullet were excessive, and that better results could be obtained by reducing its muzzle velocity to, say, 2000 f. s., and either *increasing* the weight of the bullet or *diminish-*

ing its calibre. We will endeavor to show this by the following examples:

Example 2. How much would the weight of the Lebel bullet have to be increased in order that, for a range of 1000 yards, the maximum ordinate should still be 27.6 feet, with a muzzle velocity of but 2000 f. s.?

This example is solved like the preceding, the only difference being that, in this case, we assume a value for u instead of V. We shall find, in this case,

$$\begin{aligned} u = v &= 850 \text{ f. s.}; \\ \log C &= 9.59062; \\ \phi &= 1^\circ\ 32'\ 52''; \\ \omega &= 2^\circ\ 39'; \\ y_0 &= 27.6 \text{ feet}; \\ w &= 268.89 \text{ grains.} \end{aligned}$$

The weight of the bullet would have to be increased, therefore, 37.4 grains, or about 16 per cent. This would increase the length of the bullet about one-half of a calibre.

It will be seen by comparing the results obtained that the trajectory of the hypothetical bullet is not quite so flat in the ascending branch, but flatter in the descending branch than that of the Lebel bullet for a range of 1000 yards; and therefore, *for this range*, the danger-space is slightly increased by increasing the weight of the bullet and diminishing the muzzle velocity, while the striking velocity is increased by 30 f. s.

Example 3. How much would the calibre of the Lebel rifle have to be diminished in order that for a range of 1000 yards the maximum ordinate should still be 27.6 feet, with a muzzle velocity of 2000 f. s.?

We have here $w = 231.48$ grains and $\log C = 9.59062$, to determine d by the formula

$$d = \left(\frac{w}{7000C}\right)^{\frac{1}{2}}.$$

$$
\begin{aligned}
\log C &= 9.59062 \\
\log 7000 &= 3.84510 \\
\hline
& 3.43572 \\
\log w &= 2.36451 \\
\hline
& 2)8.92879 \\
\hline
\log d &= 9.46439 \qquad \therefore\ d = 0.291 \text{ inch.}
\end{aligned}
$$

That is, if the calibre of the Lebel rifle be reduced from 0.314 in. to 0.291 in., the weight of bullet remaining the same, the muzzle velocity can be reduced 200 f. s. without impairing its ballistic qualities for a range of 1000 yards. The trajectory described will be, at least theoretically, precisely the same as that of Ex. 2, since the muzzle velocity and ballistic coefficient remain the same. This calibre is almost the same as that of the Hebler rifle. It will be shown in the next problem that the length of this modified bullet is very nearly four calibres.

Relation between Velocity, Weight of Projectile, and Powder Charge.—We have seen that by increasing the length of the Lebel bullet about half a calibre and diminishing the muzzle velocity by 200 f. s. we increase its ballistic qualities for a range of 1000 yards; and, also, that the same may be effected by diminishing the calibre of the rifle to that of the Hebler rifle, the weight of the projectile remaining the same. We will now see what effect these changes will have upon the powder charge.

For this purpose we will make use of the monomial formula deduced by M. Sarrau, as modified for quick-burning powder by Ensign J. H. Glennon, Instructor in Ordnance and Gunnery, U. S. Naval Academy.* This formula is

$$v = A\left(\frac{f}{w}\right)^{\frac{1}{2}} \pi^{\frac{3}{8}}\, d^{\frac{1}{4}}\, \Delta^{\frac{1}{4}}\, u^{\frac{1}{8}}\,;$$

---

* See P. N. I., Vol. XIV, page 402.

in which

$v =$ velocity of projectile;
$A =$ constant depending upon kind of powder;
$f =$ force of powder used;
$w =$ weight of projectile;
$\pi =$ weight of charge;
$d =$ calibre;
$\Delta =$ density of loading;
$u =$ distance travelled by projectile in the bore.

As our object here is simply to compare the charges necessary to give projectiles of the same calibre, but varying in weight, given velocities when fired from the same gun and under similar circumstances, it will be better to simplify the above formula as follows:

Let V_1 be the muzzle velocity developed by a charge π_1 upon a projectile whose weight is w_1, using a certain gun. Also let V_2, π_2 and w_2 be other similar quantities referring to the same gun. Then by division we have

$$\frac{V_1}{V_2} = \left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{8}} \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}.$$

$$\therefore\ V_1 = V_2 \left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{8}} \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}};$$

$$\pi_1 = \pi_2 \left(\frac{V_1}{V_2}\right)^{\frac{8}{3}} \left(\frac{w_1}{w_2}\right)^{\frac{4}{3}};$$

$$w_1 = w_2 \left(\frac{V_2}{V_1}\right)^{2} \left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{4}}.$$

If the charge remain constant,

$$V_1 = V_2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}; \qquad w_1 = w_2 \left(\frac{V_2}{V_1}\right)^{2}.$$

If the weight of shot remain constant,

$$V_1 = V_2\left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{8}}; \qquad \pi_1 = \pi_2\left(\frac{V_1}{V_2}\right)^{\frac{8}{3}}.$$

If the velocity remain constant,

$$\pi_1 = \pi_2\left(\frac{w_1}{w_2}\right)^{\frac{4}{3}}; \qquad w_1 = w_2\left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{4}}.$$

In Ex. 2 we have the following data from which to compute $\frac{\pi_1}{\pi_2}$:

$$\begin{aligned} V_1 &= 2000 \\ V_2 &= 2200 \\ w_1 &= 268.89 \\ w_2 &= 231.48 \end{aligned} \qquad \therefore \frac{V_1}{V_2} = \frac{10}{11}$$

$$\therefore \pi_1 = \left(\frac{10}{11}\right)^{\frac{8}{3}}\left(\frac{268.89}{231.48}\right)^{\frac{4}{3}}\pi_2.$$

The work by logarithms is as follows:

$$\begin{aligned} \log 10 &= 1.00000 \\ \log 11 &= 1.04139 \\ \hline & 9.95861 \times \tfrac{8}{3} = 9.88963 \end{aligned}$$

$$\begin{aligned} \log 268.89 &= 2.42957 \\ \log 231.48 &= 2.36451 \\ \hline & 0.06506 \times \tfrac{4}{3} = 0.08675 \\ \hline & \log 0.947 = 9.97638 \end{aligned}$$

$$\therefore \pi_1 = 0.947\pi_2.$$

That is, the hypothetical bullet of Ex. 2 would require about 5 per cent less powder to propel it than the actual bullet. In

other words, superior ballistic qualities are secured with a saving of ammunition.

In Ex. 3 the weights of the projectiles are the same, while the calibres vary. Therefore we have in this case, from Sarrau's formula,

$$\pi_1 = \pi_2 \left(\frac{V_1}{V_2}\right)^{\frac{8}{3}} \left(\frac{d_2}{d_1}\right)^{\frac{2}{3}}.$$

We have $V_1 = 2000$ f. s., $V_2 = 2200$ f. s., $d_1 = 0.291$ in., and $d_2 = 0.314$ in. Therefore

$$\begin{aligned} \log d_2 &= 9.49693 \\ \log d_1 &= 9.46439 \\ \hline 0.03254 \times \tfrac{2}{3} &= 0.02169 \\ \frac{8}{3} \log \frac{V_1}{V_2} &= 9.88963 \\ \hline \log 0.815 &= 9.91132 \end{aligned}$$

$$\therefore \pi_1 = 0.815\pi_2,$$

which is a still greater saving of powder.

We have assumed in this discussion that the capacity of the chamber varies with the charge in such a way that the density of loading remains constant.

PROBLEM XVIII.

To calculate the volumes and weights of oblong projectiles, and their ballistic coefficients.

CASE I. *Solid Shot.* We will suppose the solid shot to be a right cylinder terminated at the forward end by a complete ogival head. Let d be the diameter of the cylindrical part of the shot, nd the radius of the ogive, and Ld the total length of shot, including head. The calibre of the gun is often taken for d, and the projectile is said to be L *calibres* long, and the ogive to be struck with a radius of n *calibres.* This is a very convenient unit, and we shall generally make use of it in what follows: though the diameter of the cylindrical part of an oblong projectile is never exactly equal to the calibre of the gun for which it is intended, and usually differs slightly from the diameter of the base of the head.

It may be shown by the Integral Calculus that the volume of an oblong solid shot, as defined above, is given by the following equation:

$$Vol. = \frac{\pi d^3}{4}(L - B).$$

In this equation $\frac{\pi d^3}{4}L$ is the volume of a right cylinder whose diameter is d, and length Ld; while $\frac{\pi d^3}{4}B$ is the volume of the cylinder circumscribing the head, less the volume of the head. Therefore $L - B$ is the length, in calibres, of a right cylinder of diameter d, whose volume is the same as that of the projectile. This may be appropriately called the *reduced length* of the projectile, in *calibres*, since it reduces the composite body

to a right cylinder. Calling the reduced length l, that is, making

$$L - B = l,$$

we have

$$Vol. = \frac{\pi d^3}{4} l.$$

The expression for B in terms of n is

$$B = 2n^2(2n - 1) \sin^{-1} \frac{\sqrt{4n - 1}}{2n} - \frac{6n^2 - 2n - 1}{3} \sqrt{4n - 1}.$$

The following table of the values of B for certain values of n will be found useful:

n	B In Calibres.	Remarks.
0.5	0.16667	Hemispherical head.
1.0	0.36234	Head of Parrott shot.
1.5	0.48874	Head of shot for 8″ converted rifle.
1.6	0.51038	
1.7	0.53116	
1.8	0.55117	
1.9	0.57049	
2.0	0.58919	Head of most modern shot.
2.5	0.67505	
3.0	0.75125	Hotchkiss steel shell.

Example 1. Compute the volume of a solid shot for the 12-inch rifle.

Here the mean value of $d = 11.95$ inches; $L = 3$ calibres; $B = 0.589191$; and $l = 3 - 0.589191 = 2.410809$. Therefore

$$\begin{aligned} 3 \log d &= 3.23211 \\ \log \frac{\pi}{4} &= 9.89509 \\ \log l &= 0.38216 \\ \hline \log Vol. &= 3.50936 \quad \therefore Vol. = 3231.2 \text{ cubic inches.} \end{aligned}$$

The weight of the projectile can be determined by multiplying the weight of one cubic inch of the material of which it is made, by the number of cubic inches in the projectile, as given by the above formula. The mean weight of a cubic inch of the cast-iron, of which the shot for our rifled guns is made, is 0.261 pounds. We therefore have for the weight of the above shot

$$w = 3231.2 \times 0.261 = 843.34 \text{ lbs.}$$

CASE 2. *Cored Shot.* For cored shot the reduced length is less than for solid shot by the length of cylinder whose volume is the same as the core. Call this last-named length B'. Then we shall have

$$B' = \text{volume of core} \times \frac{4}{\pi d^3},$$

and the reduced length now becomes

$$l = L - B - B'.$$

The value of B' is not constant, but decreases with the calibre. We may take for a mean value, sufficiently accurate for our purpose,

$$B' = 0.171;$$

and therefore we have for the reduced length of cored shot, having heads struck with radii of 1½ calibres,

$$l = L - 0.66,$$

and for heads struck with radii of 2 calibres,

$$l = L - 0.76.$$

Example 2. The weight of a cored shot for the 8-inch B. L. rifle is 290 lbs. What is its length?

Here $w = 290$ lbs.; $d = 8$ in.; $n = 2$; to find L.

We have, since

$$\text{weight} = \text{volume} \times 0.261,$$

$$w = \frac{\pi d^2 l}{4} \times 0.261;$$

$$\therefore l = \frac{4}{0.261\pi} \frac{w}{d^3} = 4.878 \frac{w}{d^3};$$

and therefore

$$L = 0.76 + 4.878 \times \tfrac{290}{512} = 3.52 \text{ calibres.}$$

Example 3. What would be the length of a cored shot for the 12-inch B. L. rifle, which should weigh 1500 lbs.?

In this case

$$L = 0.76 + 4.878 \times \tfrac{1500}{1728} = 4.99 \text{ calibres.}$$

Example 4. Required the weight of a cored shot for the 12-inch rifle.

We have $d = 11.95$ inches; $L = 3$ calibres; $l = 3 - 0.76 = 2.24$, and $w = 0.261 \frac{\pi}{4} l d^3 = 0.205 l d^3$.

$$\begin{aligned} 3 \log d &= 3.23211 \\ \log l &= 0.35025 \\ \log 0.205 &= 9.31173 \\ \hline \log w &= 2.89409 \qquad \therefore w = 783.6 \text{ pounds.} \end{aligned}$$

CASE 3. *Bullets for Small Arms.* The reduced length for small-arm bullets may be found, approximately, by the equation

$$l = L - 0.37.$$

Relative Weights of Oblong Projectiles.—Let w_1 be the weight of an oblong projectile having an ogival head struck with a radius of n calibres, l_1 its reduced length, and d_1 its diameter. Also, let w_2, d_2 and l_2 be the weight, diameter and

reduced length, respectively, of another oblong projectile made of the same kind of material. Then as, under the imposed condition, the weights of the two projectiles are proportional to their volumes, we have, from what has preceded, the equal ratios

$$\frac{Vol_1}{Vol_2} = \frac{w_1}{w_2} = \frac{d_1^3 l_1}{d_2^3 l_2}.$$

Therefore we have the proportion

$$w_1 : w_2 :: d_1^3 l_1 : d_2^3 l_2.$$

That is, the weights of oblong projectiles are proportional to the products of the cubes of their diameters by their reduced lengths.

If the diameters of the two projectiles are equal, we have the proportion

$$w_1 : w_2 :: l_1 : l_3.$$

That is, the weights of two oblong projectiles of the same calibre are proportional to their reduced lengths.

Ballistic Coefficients of different Oblong Projectiles.—Let C_1 and C_2, respectively, be the ballistic coefficients of two oblong projectiles. Then, by definition,

$$C_1 = \frac{w_1}{d_1^2};$$

$$C_2 = \frac{w_2}{d_2^2}.$$

Therefore, by division,

$$\frac{w_1}{w_2} = \frac{d_1^2 C_1}{d_2^2 C_2} = \frac{d_1^3 l_1}{d_2^3 l_2}.$$

From this we deduce the proportion

$$C_1 : C_2 :: d_1 l_1 : d_2 l_2.$$

That is, the ballistic coefficients of two oblong projectiles are proportional to the products of their diameters and reduced lengths.

If the diameters of the two projectiles are the same, this last proportion becomes

$$C_1 : C_2 :: l_1 : l_2 :: w_1 : w_2 .$$

That is, the ballistic coefficients of two oblong projectiles of the same calibre are proportional to their reduced lengths, and also to their weights.

If it be a condition that the ballistic efficiency of the two projectiles for the same velocity shall be the same, that is, if $C_1 = C_2$, the above proportion may be written

$$d_1 : d_2 :: l_2 : l_1 .$$

That is, if the ballistic coefficients of two oblong projectiles are the same, their diameters are inversely proportional to their reduced lengths.

We also have by definition, when $C_1 = C_2$, whatever may be the reduced lengths of the projectiles, the proportion

$$w_1 : w_2 :: d_1^2 : d_2^2 ,$$

which, combined with the proportion above, gives

$$w_1 : w_2 :: l_2^2 : l_1^2 .$$

That is, if the ballistic coefficients of two oblong projectiles are the same, their weights are inversely proportional to the square of their reduced lengths.

Example 5. The weight of a cored shot intended for a 10-inch B. L. rifle weighs 450 pounds, is 3.02 calibres in length, and has an ogival head struck with a radius of 2 calibres. From these data compute the weight of an 8-inch cored shot for the M. L. converted rifle, length 2.43 calibres, and an ogival head of $1\frac{1}{2}$ calibres.

We have $l_2 = 3.02 - 0.76 = 2.26 =$ reduced length for the 10-inch shot. $l_1 = 2.43 - 0.66 = 1.77 =$ reduced length for

the 8-inch shot. Also $d_2 = 10$, $w_2 = 450$, and $d_1 = 8$, to compute w_1 from the formula, given above,

$$w_1 = w_2 \frac{d_1^3 l_1}{d_2^3 l_2}.$$

The calculation is as follows:

$$\begin{aligned} \log w_2 &= 2.65321 \\ 3 \log d_1 &= 2.70927 \\ \log l_1 &= 0.24797 \\ \text{a. c. } 3 \log d_2 &= 7.00000 \\ \text{a. c. } \log l_2 &= 9.64589 \\ \hline \log w_1 &= 2.25634 \qquad \therefore w_1 = 180.4 \text{ pounds,} \end{aligned}$$

which agrees with the actual weight.

Example 6. A solid projectile manufactured for the Krupp 40-cm. gun is 2.8 calibres long and weighs 775 kg. Calculate the weight of a solid shot 2.5 calibres long intended for a 28.3-cm. gun. Both projectiles have ogival heads struck with radii of 2 calibres.

Here $l_2 = 2.8 - 0.5892 = 2.2108$; $l_1 = 2.5 - 0.5892 = 1.9108$; $d_2 = 40$; $w_2 = 775$; and $d_1 = 28.3$.

NOTE.—It will be observed that no notice need be taken of the units of weight and length provided they are the same for both projectiles; since they divide out in our formulas.

$$\begin{aligned} \log w_2 &= 2.88930 \\ 3 \log d_1 &= 4.35537 \\ \log l_1 &= 0.28122 \\ \text{a. c. } 3 \log d_2 &= 5.19382 \\ \text{a. c. } \log l_2 &= 9.65545 \\ \hline \log w_1 &= 2.37516 \qquad \therefore w_1 = 237.2 \text{ kg.} \end{aligned}$$

This is very nearly the average weight of these projectiles. Our formulas are rigidly true for ideal forms and identical material only. Moreover we have taken the calibre of the gun

for d, which is always a little greater than its true value. Nevertheless the results arrived at are close approximations, and enable us to calculate, in advance of fabrication, the weights of experimental projectiles and their ballistic capabilities for any proposed gun, with sufficient accuracy for most purposes.

Example 7. From the data pertaining to the 10″ cored shot as given in Ex. 5, compute the length of a cored shot for the 12-inch B. L. rifle so that its weight shall be 800 pounds.

We have $l_2 = 2.26$, $d_2 = 10$, $w_2 = 450$, $d_1 = 12$ and $w_1 = 800$, to find l_1. Solving the formula last used, for l_1, we have

$$l_1 = l_2 \frac{w_1 d_2^3}{w_2 d_1^3},$$

from which we compute l_1 as follows:

$$\begin{aligned} \log l_2 &= 0.35411 \\ \log w_1 &= 2.90309 \\ 3 \log d_2 &= 3.00000 \\ \text{a. c. } 3 \log d_1 &= 6.76245 \\ \text{a. c. } \log w_2 &= 7.34679 \\ \hline \log l_1 &= 0.36644 \quad \therefore l_1 = 2.33 = L - 0.76 \end{aligned}$$

$$\therefore L = 3.09.$$

That is, the projectile will be 3.09 calibres long. If it be a condition that the projectile shall be but 3 calibres long, we must determine the value of B which will give this length, without changing the reduced length as found above. We therefore have, to find B,

$$2.33 = 3 - B - 0.17; \qquad \therefore B = 0.5.$$

By interpolation we find from the table of the values of B, that when $B = 0.5$, $n = 1.552$. That is, to keep the projectile 3 calibres long and still have it weigh 800 pounds, the head must be struck with a radius of 1.552 calibres instead of with 2 calibres.

Example 8. Required the weight and length of a bullet for a 0.32-inch calibre rifle, which shall have the same ballistic co-

efficient as the Springfield rifle bullet, when made of the same material.

The Springfield rifle fires a bullet 2.8 calibres long and weighing 500 grains.

We therefore have $d_2 = 0.45$, $d_1 = 0.32$, $l_2 = 2.8 - 0.37 = 2.43$ and $w_2 = 500$, to compute w_1 and L_1. We have in this case the proportion

$$d_1 : d_2 :: l_2 : l_1,$$

whence

$$l_1 = l_2 \frac{d_2}{d_1} = 2.43 \times \frac{45}{32} = 3.42;$$

$$\therefore\ L_1 = 3.42 + 0.37 = 3.79.$$

That is, the new bullet will be 3.79 calibres long; or, $3.79 \times 0.32 = 1.21$ inches.

To determine the weight we have the proportion

$$w_1 : w_2 :: d_1^2 : d_2^2,$$

whence

$$w_1 = w_2 \frac{d_1^2}{d_2^2} = 500 \times \frac{1024}{2025} = 252.84 \text{ grains.}$$

That is to say, a bullet 0.32 inches calibre, weighing 252.84 grains, has the same capacity for overcoming the resistance of the air as one 0.45 inch calibre weighing 500 grains; and the two bullets would describe similar trajectories with the same muzzle velocity.

Example 9. Compare the striking energies of the two bullets in Ex. 6, supposing them to be fired with the same muzzle velocity, and therefore having the same striking velocity.

We have for the energy of the two bullets the expressions

$$E_1 = \frac{w_1 v^2}{2g}$$

and

$$E_2 = \frac{w_2 v^2}{2g},$$

and therefore

$$E_1 : E_2 :: w_1 : w_2.$$

That is, if the two projectiles have the same ballistic coefficients and are fired with the same muzzle velocity, the striking energies will be proportional to their weights. Therefore,

$$E_1 = \frac{w_1}{w_2}E_2 = \frac{252.84}{500}E_2 = 0.5057E_2;$$

that is, the striking energy of the Springfield rifle bullet will be nearly double that of the hypothetical bullet for all ranges.

Example 10. Compare the charges of powder required to give the two bullets of Ex. 8 the same muzzle velocity.

We have from Sarrau's monomial formula, when the muzzle velocities are the same,

$$\pi_1 = \pi_2 \left(\frac{d_2}{d_1}\right)^{\frac{2}{3}}\left(\frac{w_1}{w_2}\right)^{\frac{4}{3}}.$$

But when the ballistic coefficients of the two bullets are the same, we have

$$\frac{d_2}{d_1} = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}; \qquad \therefore \left(\frac{d_2}{d_1}\right)^{\frac{2}{3}}\left(\frac{w_1}{w_2}\right)^{\frac{4}{3}} = \frac{w}{w_2}.$$

We therefore have the proportion

$$\pi_1 : \pi_2 :: w_1 : w_2.$$

That is, if two bullets have their calibres and weights so proportioned that their ballistic coefficients are the same, then the charges of powder necessary to give the bullets the same muzzle velocity are proportional to the weights of the bullets, and therefore proportional to their respective striking energies, as shown above.

Example 11. If the calibre of the Lebel rifle be reduced from 0.314 in. to 0.291 in., what will be the length of the new bullet upon the supposition that the weights and ballistic coefficients of the two bullets are respectively the same? (See page 148.)

In this case we have the proportion

$$d_1 : d_2 :: l_2 : l_1;$$

$$\therefore l_1 = \frac{d_2}{d_1}l_2 = \frac{314}{221}\,l_2 = 1.08l_2.$$

If the Lebel bullet is 3.75 calibres long, we shall have

$$l_2 = 3.75 - 0.37 = 3.38;$$
$$\therefore\ l_1 = 1.08 \times 3.38 = 3.65;$$
$$\therefore\ L_1 = 3.65 + 0.37 = 4.02 \text{ calibres.}$$

Example 12. Show that by increasing the length of a cored shot of the modern type from 3 calibres to $3\frac{1}{2}$ calibres, its striking velocity for any given range may be reduced about 10 per cent without diminishing either its striking or penetrating energy.

Let w_1 and v_1 be the weight and striking velocity, respectively, of a cored shot 3 calibres long, and w_2 and v_2 the same for a $3\frac{1}{2}$-calibre shot.

Then, since the two projectiles are of the same diameter, and are assumed to have the same striking and penetrating energy, we have the following relation between their weights and striking velocities (see page 34):

$$\frac{v_2}{v_1} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}.$$

But the weights of two oblong projectiles of the same diameter are directly proportional to their reduced lengths. Therefore

$$\frac{v_2}{v_1} = \left(\frac{l_1}{l_2}\right)^{\frac{1}{2}} = \left(\frac{L_1 - 0.76}{L_2 - 0.76}\right)^{\frac{1}{2}} = \left(\frac{2.24}{2.74}\right)^{\frac{1}{2}};$$
$$\therefore\ v_2 = 0.9042 v_1.$$

Example 13. With the conditions of Ex. 12, suppose the muzzle velocity of the 3-calibre shot to be 2100 f. s., and the range such that the striking velocity is 1400 f. s. What would be the muzzle velocity of the $3\frac{1}{2}$-calibre shot for the same range?

By Problem II we have for the two shots the following equations:

$$S(v_1) - S(V_1) = \frac{X}{C_1},$$
$$S(v_2) - S(V_2) = \frac{X}{C_2};$$

whence, by division,

$$\frac{S(v_2) - S(V_2)}{S(v_1) - S(V_1)} = \frac{C_1}{C_2}.$$

But the ballistic coefficients of two oblong projectiles of the same diameter are proportional to their reduced lengths. Therefore

$$S(V_2) = S(v_2) - \frac{l_1}{l_2}\{S(v_1) - S(V_1)\}.$$

We have found in Ex. 12 that

$$v_2 = 0.9042 v_1 = 0.9042 \times 1400 = 1265.88 \text{ f. s.},$$

and

$$\frac{l_1}{l_2} = \frac{2.24}{2.74} = 0.8175.$$

Therefore we have

$$\begin{aligned}
S(v_1) &= 4878.6 \\
S(V_1) &= \underline{2024.8} \\
\log 2853.8 &= 3.45542 \\
\log 0.8175 &= \underline{9.91250} \\
\log 2333.0 &= 3.36792 \\
S(v_2) &= \underline{5589.8} \\
S(V_2) &= 3256.8 \qquad \therefore\ V_2 = 1763 \text{ f. s.}
\end{aligned}$$

We see from this that by increasing the length of the shot one half a calibre, the muzzle velocity may be reduced 337 f. s., and still have the same striking energy for the given range. The range required to reduce the velocity from 2100 f. s. to 1400 f. s. would, of course, depend upon the diameter and weight of the projectile—in other words, upon the value of C. For the new 8-inch navy gun, for example, the range would be 4129 yards; while for the 6-inch navy gun it would be 2936 yards.

Example 14. With the data of Ex. 13 deduce the relative charges of powder required for the two projectiles.

To solve this example we will make use of Sarrau's monomial formula for *slow-burning* powder, viz.,

$$v = M\frac{\pi^{\frac{3}{8}}\varDelta^{\frac{1}{4}}d^{\frac{1}{8}}u^{\frac{3}{16}}}{w^{\frac{7}{16}}}.*$$

For the same powder and gun, and assuming the density of loading to be the same for both projectiles, we have the following relation between the muzzle velocities, weights of projectiles and charges in the two cases:

$$\frac{V_1}{V_2} = \left(\frac{w_2}{w_1}\right)^{\frac{7}{16}}\left(\frac{\pi_1}{\pi_2}\right)^{\frac{3}{8}};$$

which differs but very slightly from the corresponding formula for quick-burning powder given on page 148. From this formula we have

$$\pi_2 = \pi_1\left(\frac{V_2}{v_1}\right)^{\frac{8}{3}}\left(\frac{w_2}{w_1}\right)^{\frac{7}{6}}.$$

But since

$$\left(\frac{w_2}{w_1}\right)^{\frac{7}{6}} = \left(\frac{v_1}{v_2}\right)^{\frac{7}{3}}, \quad \text{(Ex. 12)}$$

we have

$$\pi_2 = \pi_1\left(\frac{V_2}{V_1}\right)^{\frac{8}{3}}\left(\frac{v_1}{v_2}\right)^{\frac{7}{3}}.$$

In our example we have

$$\frac{V_2}{V_1} = \frac{1763}{2100}$$

and

$$\frac{v_1}{v_2} = \frac{1}{0.9042},$$

whence

$$\begin{aligned} \tfrac{8}{3}\log\frac{1763}{2100} &= 9.79742 \\ \tfrac{7}{3}\log 0.9042 &= \underline{9.89795} \\ \log 0.7934 &= 9.89947 \\ \therefore\ \pi_2 &= 0.7934\pi_1. \end{aligned}$$

* For the definitions of the symbols see page 149.

The powder charge can, therefore, be reduced 20 per cent without diminishing the striking energy.

Example 15. Determine the relative maximum pressures in the gun with the two charges of Exs. 13 and 14.

Sarrau's formula for the maximum pressure upon the base of the projectile is

$$P = Kd^{-2}\Delta w^{\frac{1}{2}}\pi^{\frac{1}{2}};$$

that is, the maximum pressure for the same powder is directly proportional to the square root of the product of the weight of projectile and weight of charge, to the density of loading, and, inversely, to the square of the calibre. For the same gun d is constant; and as the capacity of the powder-chamber may be imagined to diminish as the charge becomes smaller, we may also regard Δ as constant. We therefore have for the solution of our example the equation

$$P_2 = P_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}\left(\frac{\pi_2}{\pi_1}\right)^{\frac{1}{2}} = P_1\left(\frac{v_1}{v_2}\right)\left(\frac{\pi_2}{\pi_1}\right)^{\frac{1}{2}};$$

$$\therefore\ P_2 = P_1\left(\frac{\pi_2}{\pi_1}\right)^{\frac{1}{2}} \div \frac{v_2}{v_1};$$

$$\therefore\ P_2 = \frac{\sqrt{0.7934}}{0.9042}P_1 = 0.9851P_1.$$

The pressure upon the base of the heavier projectile is, therefore, slightly less than upon the lighter one; and we may, in consequence, fairly assume that the same is true with reference to the walls of the gun.

The calculations of the last four examples have been made for long fighting ranges, viz., 4000 yards for 8-inch and 3000 yards for 6-inch guns. For these and *still longer* ranges the calculations show that a gun which fires a $3\frac{1}{2}$-calibre shot with a muzzle velocity of about 1750 f. s. has the same efficiency for penetrating armor as a similar gun firing a 3-calibre shot with a muzzle velocity of 2100 f. s.; with a saving of 20 per cent of powder and with a less pressure upon the walls of the gun. Moreover, the trajectory of the heavier projectile is flatter

for the same range than that of the lighter projectile; and, therefore, more likely to hit the object aimed at.

For ranges *less* than those given above the advantages of the heavier projectile over the lighter are less marked than for the longer ranges; but they still exist. The pressure upon the walls of the gun may become a little greater for *short* ranges with the heavier than with the lighter projectile, but not enough greater to be of any consequence.

Example 16. Our 12-inch B. L. rifle, with a charge of 265 pounds of powder, gives to a cored shot 3 calibres long, and having an ogive of 2 calibres, a muzzle velocity of 1800 f. s. What charge of the same kind of powder would be necessary to give to a similar projectile 5 calibres long a muzzle velocity of 1915 f. s.? Also, what would be the maximum pressure in the gun?

In this example, as in the preceding, we assume that the powder chamber is enlarged as the charge increases in such a way that the density of loading remains constant.

We have $V_1 = 1800$, $V_2 = 1915$, $L_1 = 3$, $L_2 = 5$, $\pi_1 = 265$ and $P_1 = 34000$, to find π_2 and P_2. We first find $l_1 = 3 - 0.76 = 2.24$, and $l_2 = 5 - 0.76 = 4.24$.

For the charge we have

$$\pi_2 = \pi_1\left(\frac{V_2}{V_1}\right)^{\frac{8}{3}}\left(\frac{w_2}{w_1}\right)^{\frac{7}{6}},$$

in which, for the ratio of the weights of the projectiles, we may substitute the ratio of their reduced lengths. We therefore have

$$\pi_2 = 265\left(\frac{1915}{1800}\right)^{\frac{8}{3}}\left(\frac{4.24}{2.24}\right)^{\frac{7}{6}}$$

$$= 265 \times 1.1796 \times 2.1053 = 658.07 \text{ lbs.},$$

which is the charge required.

For the maximum pressure we have

$$P_2 = P_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}\left(\frac{\pi_2}{\pi_1}\right)^{\frac{1}{2}};$$

$$\therefore\ P_2 = 34000\left(\frac{4.24}{2.24}\right)^{\frac{1}{2}}\left(\frac{658.07}{265}\right)^{\frac{1}{2}} = 73714 \text{ pounds.}$$

Example 17. A 9-pound shell fired from a 3-inch M. L. rifle with a charge of 2 pounds of I. K. powder has a muzzle velocity of 1495 f. s. With what charge should a 12-pound shrapnel be fired from the same gun at a target 578 yards off in order to have the same remaining velocity that it would have at 2500 yards if fired with the full charge of 2 pounds?

In this example we have given $d_1 = d_2 = 3$ in., $w_1 = 9$ lbs., $w_2 = 12$ lbs., $X_1 = 7500$ feet, $X_2 = 1734$ feet, $V_1 = 1495$ f. s. and $v_1 = v_2$, to compute first V_2, and then π_2 knowing $\pi_1 = 2$ pounds.

Proceeding as in Ex. 13, we find for the muzzle velocity

$$S(V_2) = \frac{X_1}{C_1} - \frac{X_2}{C_2} + S(V_1),$$

and for the charge

$$\pi_2 = \pi_1\left(\frac{V_2}{V_1}\right)^{\frac{8}{3}}\left(\frac{w_2}{w_1}\right)^{\frac{7}{6}}.$$

Or, in numbers,

$$S(V_2) = 7500 - \tfrac{3}{4}(1734) + S(1495) = 10616.0;$$

$$\therefore\ V_2 = 821.3 \text{ f. s.}$$

$$\pi_2 = 2\left(\frac{821.3}{1495}\right)^{\frac{8}{3}}\left(\frac{4}{3}\right)^{\frac{7}{6}} = 0.56635 \text{ pounds.}$$

Example 18. How should the fuse of the shrapnel of Ex. 17 be cut so as to burst 50 yards in front of the target?

We have here $V = 821.3$ f. s., $C = \frac{4}{3}$ and $X = 1584$ feet, to compute t.

Proceeding as in Prob. V, we find

$$S(v) = \tfrac{3}{4} \times 1584 + S(V) = 11804.0;$$

$$\therefore\ v = 765 \text{ f. s.}$$

$$t = \tfrac{4}{3}\{T(765) - T(821.3)\};$$

$$\therefore\ t = 2.0 \text{ seconds.}$$

Similar Oblong Projectiles.—Two oblong projectiles are similar when they are the same number of calibres in length and have ogival heads struck with the same number of radii. If these conditions are fulfilled, it is evident from definition

that their reduced lengths are the same; and, therefore, the proportion on page 156 becomes

$$w_1 : w_2 :: d_1^3 : d_2^3;$$

that is, the weights of similar oblong projectiles made from the same material are proportional to the cubes of their diameters. They are, in fact, similar geometrical solids; and their weights may therefore be taken proportional to the cubes of *any* two homologous lines.

We also have, when $l_1 = l_2$, the proportion

$$C_1 : C_2 :: d_1 : d_2;$$

that is, the ballistic coefficients of two similar oblong projectiles are proportional to their diameters.

Length of Ogival Head.—The length of the ogival head of a projectile, in calibres, is given by the formula

$$\text{Length of head} = \tfrac{1}{2}\sqrt{4n - 1}.$$

The length of the cylindrical part is, therefore, in calibres,

$$\text{Length of cylinder} = L - \tfrac{1}{2}\sqrt{4n - 1}.$$

Multiplying these results by d gives the respective lengths in inches.

Applying these formulas to the data of Ex. 7, we find, when $L = 3.09$ and $n = 2$,

Length of head = 15.87 inches
Length of cylinder = 21.21 "

Length of projectile = 37.08 inches.

If $L = 3$ and $n = 1.552$, we find

Length of head = 14.71 inches
Length of cylinder = 21.29 "

Length of projectile = 36.00 inches.

PROBLEM XIX.

Given the muzzle velocity (V), *the angle of departure* (ϕ), *and the range* (X), *to compute the ballistic coefficient* (C), *and coefficient of reduction* (c).

Solution. Eliminating C from (8) and (15), and writing for convenience, u for u_ω, we have

$$\sin 2\phi = X \left\{ \frac{I(u_0) - I(V)}{S(u) - S(V)} \right\};$$

and, by (14),

$$I(u_0) = \frac{A(u) - A(V)}{S(u) - S(V)}.$$

From these two equations we must find u by trial; and then C and c by the equations

$$C = \frac{X}{S(u) - S(V)},$$

and

$$c = \frac{\delta_1}{\delta} \frac{w}{Cd^2}.$$

Example 1. Firing at Meppen with a 20.93-cm. gun, the observed range for an angle of departure of 5° 38′, and muzzle velocity of 1709.35 f. s., was 13441.8 feet. From this data determine the values of C and c.

For this example we have $d = 20.93$ cm., $w = 140$ kg., $V = 1709.35$ f. s., $\phi = 5° \, 38'$, $X = 13441.8$ feet, and $\frac{\delta_1}{\delta} = 0.9781$.

An approximate value of C may be computed from the given data by omitting the factor c; and then an approximate value of u by the equation

$$S(u) = \frac{X}{C} + S(V).$$

From this preliminary calculation we find $u = 1126$; and as this is less than its true value on account of having taken C too small, we will assume for a first approximation $u = 1150$. The operations are as follows:

$$
\begin{aligned}
S(u) &= 6321.8 \\
S(V) &= 3473.5 \\
\hline
S(u) - S(V) &= 2848.3
\end{aligned}
$$

$$
\begin{aligned}
A(u) &= 328.27 \\
A(V) &= 70.74 \\
\hline
\log 257.53 &= 2.41083 \\
\log 2848.3 &= 3.45459 \\
\hline
\log 0.09042 &= 8.95624 = \log I(u_0) \\
I(V) &= 0.04860 \\
\hline
\log 0.04182 &= 8.62138 \\
\log X &= 4.12846 \\
\text{a. c. } \log 2838.3 &= 6.54541 \\
\hline
\log \sin 2\phi &= 9.29525
\end{aligned}
$$

The real value of $\log \sin 2\phi$ is 9.29087; $\quad \therefore\ e_1 = -0.00438$.

Next assume $u = 1170$, and it will be found that

$$e_2 = +0.00202.$$

We therefore have the proportion

$438 + 202 : 20 :: 202 : 6.3$; $\quad \therefore\ u = 1170 - 6.3 = 1163.7$.

To compute C and c we proceed as follows:

$$
\begin{aligned}
S(u) &= 6227.6 \\
S(V) &= 3473.5 \\
\hline
\log 2754.1 &= 3.43998 \\
\log X &= 4.12846 \\
\hline
\log C &= 0.68848
\end{aligned}
$$

As the expression for c contains the factor $\frac{w}{d^2}$, in which w is expressed in kilogrammes and d in centimetres, we must, to avoid the necessity of reducing them to English units, multiply by the factor

$$\frac{\text{No. of pounds in one kilogramme}}{(\text{No. of inches in one centimetre})^2}.$$

The logarithm of this factor is 1.15298;

$$\therefore\ c = [1.15298]\frac{\delta_{\prime}}{\delta}\,\frac{w}{Cd^2}.$$

We have, therefore,

$$\begin{aligned}
\log w &= 2.14613\\
\log \frac{\delta_{\prime}}{\delta} &= 9.99038\\
\text{const.}\log &= 1.15298\\
\text{a. c.}\log C &= 9.31152\\
\text{a. c.}\log d^2 &= 7.35846\\
\hline
\log c &= 9.95947\\
\therefore\ c &= 0.911
\end{aligned}$$

Example 2. Determine the values of C for different ranges, for the 3.2-inch steel B. L. rifle.

This gun was fired at Sandy Hook in March, 1885, for the purpose of determining the ranges for differences of 2° in elevation, beginning with 2° and ending with 20° elevation, the limit permitted by the carriage.

The principal characteristics of this gun, and of the ammunition used in these experiments, are as follows:

Calibre of gun,	3.2 inches
Weight of gun,	791 pounds
Length of bore,	26 calibres
Twist,	Uniform, one turn in 30 calibres
Weight of shot,	13 pounds
Radius of ogive,	$1\frac{1}{2}$ calibres
Powder charge,	$3\frac{1}{2}$ pounds, Dupont's *L. X. A.* Density 1.706 Granulation 270

Nineteen shots were fired for velocity, which gave, all reductions being made, a muzzle velocity of 1608 f. s. Twelve shots were also fired at a target 50 feet from the gun to determine the angle of jump.

The following table gives a summary of the firing, up to 10° elevation. The ranges and times of flight are each a mean of 10 shots.

The values of $\frac{\delta_{\prime}}{\delta}$ are taken from Table III with the observed barometric pressures and temperatures for arguments. The values of W_p are computed by the method given on page 40.

Elevation.	Jump.	Angle of departure (ϕ).	Mean observed range X_0 (feet).	Elevation of gun above striking point (feet).	Observed time of flight (seconds).	$\frac{\delta_{\prime}}{\delta}$	W_p (feet).
°	′ ″	° ′ ″					
2	21 00	2 21 00	4755	14.3	4.00	0.920	+11.19
4	22 15	4 22 15	7093	16.6	6.60	0.932	8.80
6	22 45	6 22 45	9109	12.0	9.00	0.942	7.46
8	23 15	8 23 15	10907	12.9	11.45	0.942	7.46
10	22 00	10 22 00	12451	12.7	13.80	0.942	7.46

In this example we will endeavor to eliminate the influence of the wind upon the ranges: that is, we will determine, at least approximately, what the ranges would have been had the atmosphere been calm during the firing. The direction of the wind-component parallel to the plane of fire (W_p) was in all cases from the target toward the gun, and therefore diminished the ranges.

As the equations of Problem VIII do not apply in this case, we will make use of the following empirical equation for computing ΔX:*

$$\Delta X = W_p \left\{ T - \frac{X}{V} \frac{a \cos \phi}{2a - 1} \right\},$$

* See *Balistique Extérieure*, by Major Muzeau of the French Artillery.

in which

$$a = \frac{V^2 \sin 2\phi}{gX}.$$

This equation gives a fair approximation for ΔX for moderate winds and ranges; but for long ranges the results are somewhat too small, as the X in the second member, and in the expression for a, should be the range in an undisturbed atmosphere, whereas we necessarily use the observed range.

The following is an example of the computation of ΔX by the above formula:

We have $V = 1608$, $\phi = 2^\circ\ 21'$, $X = 4755$ and $T = 4$.

$$\begin{aligned}
\log V^2 &= 6.41257 \\
\log \sin 2\phi &= 8.91349 \\
\text{a. c.} \log g &= 8.49268 \\
\text{a. c.} \log X &= 6.32285 \\
\hline
\log a &= 0.14159 \qquad \therefore a = 1.3855 \\
& \qquad\qquad 2a - 1 = 1.7710 \\
\log X &= 3.67715 \\
\log \cos \phi &= 9.99963 \\
\text{a. c.} \log V &= 6.79371 \\
\text{a. c.} \log (2a - 1) &= 9.75178 \\
\hline
\log 2.31 &= 0.36386 \\
T &= 4.00 \\
\hline
1.69 \times 11.19 &= 19 \text{ feet} = \Delta X.
\end{aligned}$$

The corrected range is therefore

$$X = 4755 + 19 = 4774 \text{ feet}.$$

By the principle of the *rigidity of the trajectory* we may consider the observed ranges (corrected for wind as above) horizontal, provided we increase the angles of projection by the corresponding angles of depression of the point of fall below the level of the gun. That is, the new angle of projection will be determined by the equation

$$\phi' = \phi + \epsilon. \quad \text{(See page 124.)}$$

The following table gives the angles of departure, upon the supposition that the ranges are horizontal; and also the observed ranges corrected for wind:

Elevation.	Angle of depression (ϵ).	Angle of departure for horizontal range (ϕ').	ΔX (feet).	Corrected (range feet).
	° ′ ″	° ′ ″		
2	0 10 18	2 31 18	19	4774
4	8 01	4 30 16	31	7124
6	4 30	6 27 15	39	9148
8	4 03	8 27 18	53	10960
10	3 29	10 25 29	67	12518

The following is the computation of C for the angle of elevation of 2°:

Assume for a first approximation, $u = 990$.

$$S(u) = 7852.5$$
$$S(V) = 3903.7$$
$$z = 3948.8$$

$$I(u) = 614.16$$
$$A(V) = 93.77$$
$$\log 520.39 = 2.71633$$
$$\log z = 3.59647$$

$$\log 0.13178 = 9.11986$$
$$I(V) = 0.05867$$

$$\log 0.07311 = 8.86398.$$
$$\log X = 3.67888$$
$$\text{a. c. } \log z = 6.40323$$

$$\log \sin 2\phi = 8.94639$$
$$\text{True value} = 8.94403$$

$$\therefore e_1 = \quad -236$$

Next we assume $u = 1000$, and find by a similar process

$$e_2 = +485;$$

$$\therefore 236 + 485 : 10 :: 236 : 3.3;$$

$$\therefore u = 990 + 3.3 = 993.3.$$

And this value of u completely satisfies the above equations. The computation of C and c is as follows:

$$\begin{aligned} S(u) &= 7808.1 \\ S(V) &= \underline{3903.7} \\ \log 3904.4 &= 3.59155 \\ \log X &= \underline{3.67888} \\ \log C &= 0.08733 \end{aligned}$$

$$\begin{aligned} \log \frac{w}{d^2} &= 0.10364 \\ \log \frac{\delta_{\prime}}{\delta} &= 9.96379 \\ \text{a. c. } \log C &= \underline{9.91267} \\ \log c &= 9.98010 \qquad \therefore c = 0.955. \end{aligned}$$

Proceeding in the same way for the remaining angles of elevation, we have the following results:

Elevation.	u	log C	c	Computed time of flight.	Observed—computed time of flight.
°				"	"
2	993.3	0.08733	0.955	3.91	+0.09
4	865.6	0.08316	0.977	6.48	0.12
6	795.0	0.10105	0.948	8.87	0.13
8	741.8	0.11433	0.919	11.19	0.26
10	698.3	0.12217	0.903	13.38	0.42

The mean of the first three values gives, for angles of elevation from 0° to 6°, which includes all ranges up to 3000 yards,

$$c = 0.96.$$

For angles of elevation exceeding 6°, or for ranges exceeding 3000 yards,

$$c = 0.91.$$

Best Method of Computing the Ballistic Coefficient.— The most accurate method of calculating the value of c is that given in Problem IV. But where the terminal velocities cannot be measured directly by a chronoscope, the above method is as accurate and convenient as any that can be devised. We might determine the values of C and c from the observed range and time of flight, taking account of the effect of the wind, by a combination of Problems VI and VII, which by eliminating C gives

$$\frac{X \pm TW_p}{T} = \frac{S(u \pm W_p) - S(V \pm W_p)}{T(u \pm W_p) - T(V \pm W_p)},$$

from which to determine $u \pm W_p$ by trial. C would then be computed by the equation

$$C = \frac{X \pm TW_p}{S(u \pm W_p) - S(V \pm W_p)}.$$

This method requires, however, that the time of flight should be known to within one tenth of a second in order to be even approximately correct; and is, therefore, of no practical value.

PROBLEM XX.

To calculate the drift of an oblong projectile.

It is found by experiment that elongated projectiles having ogival heads, fired from rifled guns which, like those in our service, give a right-handed rotation, always deviate to the right in a calm atmosphere; while those fired from guns which give a left-handed rotation, as with the French naval guns, deviate to the left. This deviation is called *drift* (French *dérivation*). It is generally constant for the same gun and range, and can therefore be tabulated and allowed for in laying the gun.

No entirely satisfactory explanation of this difficult subject can be given without the aid of the higher mathematics.

The subject has been very fully treated by the following authors:

General Mayevski. Traité de Balistique Extérieure. Paris, 1872.

Le Comte de Sparre. Mouvement des Projectiles oblongs dans le cas du Tir de Plein Fouet. Paris, 1875.

General Mayevski. On the Solution of Problems in Direct and Curved Fire, St. Petersburg, 1882. This work is written in the Russian language. A translation into Italian of the parts relating to drift may be found in the Revista di Artiglieria e Genio for 1884, vol. 3, page 81.

Major Muzeau. Balistique Extérieure. Lithographie de l'École d'Application de l'Artillerie et du Génie, 1883.

This work was first published in the Révue d'Artillerie, vols. 12 and 13, Paris, 1879.

Lieutenant J. Baills. Traité de Balistique Rationnelle, Paris, 1883.

Major Astier. Mouvement des Projectiles oblongs.

Prof. A. G. Greenhill. On the Derivation, or Drift, of Elon-

gated Projectiles. Proceedings Royal Artillery Institution, vol. 11, page 124.

In the following attempt to explain, without the aid of mathematics, the principal phenomena connected with the subject of drift, we have derived great assistance from the fine work of Muzeau, cited above.

We shall suppose the rotation of the projectile to be from left to right in the upper hemisphere as viewed from the rear of the gun. If it should have a left-handed rotation all the phenomena would be reversed.

When an oblong projectile, properly centred in the gun, emerges from the bore, its axis sensibly coincides with the tangent to the trajectory described by its centre of gravity; and the resistance of the air acting symmetrically in lines parallel to the direction of motion has a single resultant directed along the axis of the projectile, which is also the axis of rotation. There is, therefore, nothing at first to cause the projectile to deviate from the plane of fire, or to change the direction of its axis. But, under the action of gravity, the tangent to the trajectory immediately begins to fall below the plane of its initial direction, while the rotation of the projectile, on the contrary, tends to keep its axis parallel to its original direction. The result of this slight separation of these two lines is that the resultant of the resistance of the air takes a direction oblique to both of them, making a small angle with the axis, which it cuts at a point called the centre of pressure, and which for service projectiles is always situated between the point of the projectile and its centre of gravity.

This resultant may be resolved into two components, one of which, and by far the larger of the two, acts in the direction of the axis of the projectile, and in opposition to its motion; while the other acts normal to the axis through the centre of pressure, and tends to raise the point of the projectile and cause it to revolve around an axis perpendicular to the plane of fire, in such a way that if the projectile had no motion of rotation it would "tumble," as it is called. This is sometimes observed in practice-firing with our converted 8-inch rifles, when

the projectiles "strip" or fail to take the groove. But if the projectile has a sufficient motion of rotation about its axis this effect is not produced. The upward pressure combined with the right-hand rotation causes the point of the projectile to move off slowly to the right, as may easily be verified with the gyroscope.

In the next instant the same effects are repeated except that the resultant of the resistance of the air has changed its direction of action with reference to the axis,—or, rather, the axis has changed its direction with reference to the resultant,—which latter now supplies a component whose effect is to thrust the point of the projectile to the right, and which, combined with rotation, causes the point to fall, or droop, as it is called. The constant action of these forces has for effect to cause the axis of the projectile to describe a conical surface around the tangent to the trajectory from left to right, the apex of the cone being at the projectile's centre of gravity. This motion of the axis of the projectile around the tangent is called precession from analogy with the similar motion of the earth's axis around the axis of the ecliptic.

The plane passing through the axis and the tangent, turning with the former around the tangent, the resultant resistance of the air which is always contained in this plane makes an increasing angle with the plane of fire, and furnishes a component whose effect is to move the projectile from this plane. The lateral displacement which the projectile thus suffers is called *drift.*

The angular velocity with which the axis of the projectile turns around the tangent is very small, since it is in inverse ratio of the angular velocity of the projectile, which in direct fire is very great.

It is not strictly correct to say that the axis of the projectile revolves around the tangent. The rotation of the projectile really takes place around an instantaneous axis which describes in the interior of the body a cone around the axis of the projectile, and in space another cone around the tangent. The motion of the projectile is the same as if the first cone, consid-

ered as attached to the projectile and carried along with it, rolled upon the surface of the second cone.

The effect of this is to cause any point of the axis of the projectile to take up an epicycloidal motion around the tangent, and the axis to describe a sort of corrugated cone. This motion of the axis is called *nutation*.

In direct fire, however, the instantaneous axis sensibly coincides with the axis of the projectile; and therefore the resistance of the air depends almost entirely upon the velocity of translation, as has been shown by experiment.

From the above it follows that the effects of the rotation imparted to a projectile by the rifling are—

1st. To increase the stability of the projectile by overcoming the perturbating effects of the resistance of the air which tend to upset it.

2d. To keep the axis of the projectile near the tangent (a condition very favorable to long ranges) by impressing upon the first of the two lines a motion of rotation around the second.

Both of these theoretical results are known to be true from experience and independently of any theory.

Mayevski's Formula for the Drift of an Oblong Projectile.—Mayevski, following De Sparre's method, which is founded upon the hypothesis that the angle made by the axis of the projectile with the tangent of the trajectory at any instant is very small, and which holds true for direct fire, has deduced an expression for the drift, which, when modified for direct fire, and reduced to English units, is as follows:

$$D = \frac{\pi\mu}{n}\frac{\lambda}{h}\frac{gCV}{\cos^3\phi}\left\{\frac{B(u) - B(V)}{S(u) - S(V)} - M(V)\right\}\frac{X}{10000}.$$

In this equation

$$\mu = \frac{k}{R^2},$$

in which k is the radius of gyration of the projectile with reference to its axis, and R its radius. Mayevski gives for the mean value of μ for cored shot of the modern type,

$$\mu = 0.53.$$

The method of computing μ will be given further on.

$\frac{\lambda}{h}$ is a quantity depending upon the length of the projectile, the shape of the head, the angle which the resultant resistance makes with the axis, and the distance of the centre of pressure from the centre of gravity. Mayevski gives the following mean values:

$$\frac{\lambda}{h} = 0.41 \text{ for projectiles 2.5 calibres long.}$$
$$= 0.37 \quad " \quad " \quad 2.8 \quad " \quad "$$
$$= 0.32 \quad " \quad " \quad 3.4 \quad " \quad "$$

n is the length of twist in calibres—that is, the distance the projectile advances, in calibres, while making one revolution.

$$g = \text{acceleration of gravity};$$
$$\pi = 3.1416.$$

$B(u)$, $B(V)$, and $M(V)$ are certain functions of the velocities, defined by the equations

$$M(u) = -\int \frac{du}{u^2 f(u)} \quad \text{and} \quad B(u) = -\int M(u) \frac{udu}{f(u)}.$$

Their values are given in Table I. $S(u)$ and $S(V)$ are the space functions, already well known.

Example 1. Compute a table of drift for the cored shot of the 8-inch M. L. converted rifle.

For this gun we have the following data:

$V = 1404$ f. s.; $w = 183$ lbs.; $d = 8$ in.; $c = 1$; $n = 45$; $\mu = 0.53$; $\frac{\lambda}{h} = 0.41$; $g = 32.16$. Making these substitutions, and reducing, the formula for drift for this gun becomes

$$D = 0.19586 \left\{ \frac{B(u) - B(V)}{S(u) - S(V)} - M(V) \right\} \frac{X}{\cos^2 \phi}.$$

A table of drift would, of course, form part of a range table, with the range as argument; and would be computed after u and ϕ had been determined by Problem XII. We may therefore consider these quantities known. As an example of the numerical work required, we will compute the drift for a range of 3000 yards or 9000 feet. For this range we have $u = 968.8$ and $\phi = 5° 43'$.

From Table I we have

$$S(u) = 8006.0$$
$$S(V) = 4858.5$$
$$z = 3147.5$$
$$B(u) = 72.480$$
$$B(V) = 17.381$$
$$\log 55.099 = 1.71144$$
$$\log z = 3.49797$$
$$\log 0.01751 = 8.24317$$
$$M(V) = 0.00848$$
$$\log 0.00903 = 7.95569$$
$$\text{const. } \log = 9.29195$$
$$\log X = 3.95424$$
$$\log \sec^2 \phi = 0.00650$$
$$\log D = 1.20838 \qquad \therefore D = 16.16 \text{ feet.}$$

We have computed the following range table for this gun by methods already fully explained and illustrated.

NOTE.—The computations are for the 8-inch converted rifles numbered above 28. For the guns numbered from 1 to 28 the twist of rifling is one turn in 60 calibres.

RANGE TABLE FOR THE 8-INCH CONVERTED RIFLE.

(Data taken from Ordnance Memoranda, No. 24.)

Range (yards).	ϕ	ω	Striking Velocity.	T	D yards.
	° ′	° ′		″	
500	0 44	0 47	1303	1.10	0.1
1000	1 33	1 43	1213	2.30	0.4
1500	2 27	2 50	1135	3.59	1.0
2000	3 27	4 08	1070	4.96	2.0
2500	4 32	5 38	1021	6.40	3.4
3000	5 43	7 14	982	7.92	4.4
3500	6 59	9 01	947	9.52	7.9
4000	8 21	10 58	916	11.19	11.1
4500	9 49	13 06	888	12.94	15.1
5000	11 24	15 25	862	14.78	20.1

Example 2. Compute the drift of a shell fired from the 3.2-inch B. L. steel field-gun, for a range of 2500 yards.

For this gun and range we have the following data: $V = 1608$ f. s.; $w = 13$ lbs.; $d = 3.2$ inches; $c = 0.96$; $\log C = 0.12137$; $n = 30$; $\frac{\lambda}{h} = 0.37$; and $g = 32.16$. For shell we have $\mu = 0.64$.

Applying these numbers, we have the following formula for computing the drift for this gun:

$$\text{Drift} = 0.16959 \left\{ \frac{B(u) - B(V)}{S(u) - S(V)} - M(V) \right\} \frac{X}{\cos^3 \phi}.$$

For a range of 2500 yards we have $u = 877.6$ and $\phi = 4° 40'$.

$$\begin{aligned}
B(u) &= 134.860 \\
B(V) &= \underline{10.732} \\
\log 124.128 &= 2.09387 \\
\log z &= \underline{3.75369} \\
\log 0.02189 &= 8.34018 \\
M(V) &= \underline{0.00564} \\
\log 0.01625 &= 8.21085 \\
\text{const. } \log &= 9.22939 \\
\log X &= 3.87506 \\
\log \sec^3 \phi &= \underline{0.00433} \\
\log D &= 1.31964 \\
\therefore D &= 20.88 \text{ feet} = 6.96 \text{ yards.}
\end{aligned}$$

Example 3. Compute the drift of a shell fired from the French 27-cm. gun, model of 1870–1873, for ranges from 1000 to 7000 metres.

For this gun and projectile we have the following data: $V = 505$ m. s. $= 1657$ f. s.; $w = 180$ kg.; $d = 27$ cm.; $c = 0.95$; $\log C = 0.56780$; $n = 45$; $\frac{\lambda}{h} = 0.41$; $\mu = 0.64$; and $g = 9.81$ m.

Making the necessary reductions, the formula for the drift for this gun becomes, in English units,

$$D = 0.36171 \left\{ \frac{B(u) - B(V)}{S(u) - S(V)} - M(V) \right\} \frac{X}{\cos^3 \phi}.$$

The following table gives the drift calculated by the above formula and reduced to metres; to which is added, for comparison, a column of drift taken from the official Table of Fire, which is presumably based upon observations:

Range (Metres).	Drift in Metres.	
	Calculated.	From the Table.
1000	0.4	0.4
2000	1.8	1.8
3000	4.8	4.7
4000	10.1	9.9
5000	18.5	18.1
6000	31.2	30.3
7000	49.9	47.7

Baills' Formula for Drift.—A simple approximate formula for calculating the drift of oblong projectiles has been elaborated by Lieutenant Baills of the French navy, and is given in his *Balistique Rationnelle*, page 270, Eq. (23).

This formula is, in English units,

$$D = \frac{\pi \mu g \lambda}{2nh} \cos \frac{\phi}{2} \left\{ 1 + 0.0000184 \frac{V}{C}\left(1 + \frac{2t}{3}\right) \right\} t^2.$$

It gives for the flatter trajectories of direct fire practically the same values for the drift as Mayevski's formula, with about the same labor.

Example 4. Apply Baills' formula to the data of Ex. 2.

In addition to the given data we must compute the time of flight, which we find to be 6.756 seconds.

We also find

$$\frac{\pi\mu g\lambda}{2nh}\cos\frac{\phi}{2} = 0.39842\,;$$

$$1 + 0.0000184\frac{V}{C}\left(1 + \frac{2t}{3}\right) = 1.1231\,;$$

$$t^2 = 45.643536.$$

$\therefore D = 0.39842 \times 1.1231 \times 45.643536 = 20.42$ feet, which is practically the same as that given by Mayevski's formula.

Effect of Wind upon the Drift.—In Problems VI, VII and VIII we have given formulas for determining the effects of wind upon the remaining velocity and range of an oblong projectile, which formulas are based upon the hypothesis that these effects are due to that component of the wind which is parallel to the plane of fire, and which, according to its direction, increases or diminishes the resistance the projectile encounters.

The component of the wind which is normal to the range, and which we will designate by W_n, plays a much more complicated rôle, being inextricably mixed up with the phenomena of the drift. The wind acts chiefly upon the head of the projectile, as is apparent from the fact that the air by the rapid motion of translation of the projectile through it is greatly condensed around the head, which it closely embraces, and is thrown off in stream lines, which unite again in rear of the projectile. Any sideway motion of the air will therefore cause an unequal pressure upon the head of the shot, the effect of which is to produce upon a rotating projectile a precession of the axis independent of that which causes the drift, and which greatly modifies it.

Suppose, for example, that the projectile has a right-handed rotation and that the wind blows from the left. Its action upon the head of the projectile combined with the rotation will cause the point to fall or droop, bringing its axis more nearly into coincidence with the tangent than it otherwise would be, and thus diminishes the drift. The contrary takes

place if the wind comes from the right. Let D be the drift in a calm atmosphere—or, simply, the *drift;* D_1 the diminution of the drift due to a wind blowing from the left; D_1' the increase of the drift due to a wind blowing from the right; and s the lateral displacement which the wind would give the projectile if it did not rotate; then the expressions for the deviation from the plane of fire will be: For a wind from left

$$Z = D - D_1 + s,$$

and for a wind from the right

$$Z = D + D_1' + s,$$

in which D_1' is different from D_1.

The difficulty of computing Z is still further increased by another cause. With the wind from the left, for example, the projectile, having a certain velocity of drift to the right, diminishes more and more the effect of the wind, as the difference between the sideway velocity of the projectile and that of the wind becomes less. It can easily be shown that when the time of flight is considerable, the projectile at the point of fall has generally a greater sideway velocity than the wind, this velocity being in extreme cases as much as 60 or 70 feet per second.* With the wind from the right the reverse obtains.

Didion's Method of Computing the Deviating Effect of the Wind.—The following method of determining the wind deviation independently of the drift was devised by General Didion: Suppose a velocity equal and contrary to that of the wind to be impressed upon all the elements of the system—atmosphere, projectile and gun (origin of co-ordinates),—thus producing the conditions of a calm atmosphere.

The angle which the new plane of fire makes with the primitive plane, as well as the variations ΔV and $\Delta \phi$, produced by the motion impressed upon the system, are determined, and thence the value of s. The expression for this last is, by this process,

$$s = W_n\left(T - \frac{X}{V \cos \phi}\right).\dagger$$

* Baills, *Balistique Rationnelle*, p. 319.

† Muzeau, *Balistique Extérieure*, part II. p. 121.

Maitland's Formula for Wind Deviation.—The only other method of computing s that we have seen is that first published by Colonel Maitland, R.A., in the Proceedings Royal Artillery Institution, Vol. VIII, page 348. It was subsequently reproduced in his article "Sights," in the Encyclopædia Britannica, where he gives the following formula:

$$s = W_n T - 990 \frac{w}{Ag} \log \left\{ 1 + \frac{Ag W_n T}{500w} \right\};$$

in which A, the only symbol not heretofore defined, is the area of longitudinal section of shot, in square feet. The "log" is the common logarithm, the modulus being incorporated in the factor 990.

Colonel Maitland says of this formula, that it "assumes that the wind steadily carries the shot sideways without changing the parallelism of its axis." In other words, that the deviating component of the wind W_n acts through the projectile's centre of gravity. But, as we have seen, this assumption can hardly be true. We may, however, employ an empirical equation of the same form as Colonel Maitland's, but containing a coefficient, to be determined by experiment for each kind of shot, as suggested by Lieutenant G. N. Whistler, Fifth Artillery, U. S. Army.

If p is the pressure of W_n in pounds per square foot of longitudinal section of projectile; and v the sideway velocity communicated to the projectile by this pressure, we shall have p proportional to $(W_n - v)^2$; that is, we may take

$$p \doteq \frac{(W_n - v)^2}{bg},$$

where b is a variable coefficient (here considered constant), depending partly upon the velocity of the wind, and partly upon the velocity of the projectile. Its mean value for the different kinds of guns and projectiles used in service must be determined by experiment. We therefore have for the acceleration,

$$\frac{dv}{dt} = \frac{pAg}{w} = \frac{A}{bw}(W_n - v)^2.$$

Integrating and solving with reference to v, we have

$$v = \frac{ds}{dt} = W_n - \frac{W_n}{1 + \frac{A W_n t}{bw}},$$

whence

$$s = W_n t - \frac{bw}{A} \log_e \left(1 + \frac{A W_n t}{bw}\right).$$

From this equation we must determine b by trial, having previously found the value of s by experiment.

Calculation of A.—It may be shown by the calculus that

$$A = d^2(L - B_1),$$

in which d is the diameter of the projectile in feet, L its length in calibres, while B_1 is a function of the number of calibres in the radius of the ogive (n), of the following form:

$$B_1 = \tfrac{1}{4}(2n + 1)\sqrt{4n - 1} - n^2 \sin^{-1} \frac{\sqrt{4n - 1}}{2n}.$$

The following table gives the values of B_1 for the more common values of n:

n	B_1
0.5	0.1073
1.0	0.2518
1.5	0.3437
2.0	0.4163
2.5	0.4781
3.0	0.5329

Example 5. Compute the area of the longitudinal section through the axis of an 8-inch shot for the converted rifle.

Here $L = 2.5$, $n = 1.5$ and $d = \frac{2}{3}$.

$$\therefore\ A = (2.5 - 0.3437) \times \tfrac{4}{9} = 0.9584 \text{ square feet.}$$

In order to compare Didion's and Maitland's formulas, we have computed the following table, which explains itself:

Table of deviations due to a cross-wind, of an 8-inch projectile fired from the M. L. converted rifle, for a range of 3000 yards.

$X = 9000$ feet; $V = 1404$ f. s.; $\phi = 5° 43'$; $T = 7.92$ seconds; $w = 183$ lbs., and $A = 0.9584$ sq. ft.

W_n	s (DIDION.)	s (MAITLAND.)
10	14.8 ft.	12.0 ft.
20	29.6	25.7
30	44.4	41.0
40	59.2	58.0
50	74.0	76.4

It will be seen that, for the example chosen, the results do not differ materially from each other. The actual deviations however, due to the wind, would probably be greater than those given by either method.

Twist of Rifling.—The twist of a rifled gun, whether uniform or increasing, is measured by the linear distance the projectile advances at, or near, the muzzle, while turning once about its axis. This linear distance is measured in calibres, and is, therefore, the same whether the foot or metre is the unit of length. Sometimes, especially in France, the twist is given in degrees and minutes; that is, by the angle which the grooves near the muzzle make with an element of the bore. The relation between n (number of calibres representing twist) and β (angle between groove and element) is given by the equations

$$\tan \beta = \frac{\pi}{n},$$

or

$$n = \pi \cot \beta.$$

Rotation of a Projectile about its Axis.—In treating of the motion of rotation of a projectile we will, for simplicity,

adopt the unit of length employed in motion of translation; that is, we will take d in feet instead of, as heretofore, in inches.

Revolutions per Second.—A projectile advances n calibres, or nd feet, at or near the muzzle, while making one revolution about its axis. But it also advances V feet in one second.

$$\therefore \frac{V}{nd} = \text{No. of revolutions per second.}$$

Surface Velocity of Rotation.—In one revolution, a point on the surface of a projectile passes over, in consequence of rotation, πd feet.

$$\therefore \frac{V}{nd} \times \pi d = \frac{\pi V}{n} = \text{surface velocity.}$$

Angular Velocity of Projectile's Rotation.—Since the linear velocity of any point of a projectile is proportional to its distance $\left(\frac{d}{2}\right)$ from the axis, we find the velocity of a point at unit distance (one foot), which is called angular velocity, and usually designated by ω, by the proportion

$$\frac{d}{2} : 1 :: \frac{\pi V}{n} : \omega;$$

$$\therefore \omega = \frac{2\pi V}{nd}.$$

Example 6. Compute the number of revolutions per second made by an 8-inch projectile fired from a rifled gun having a twist of one turn in 30 calibres, and which gives a muzzle velocity of 1850 f. s. Also determine its surface velocity due to rotation and its angular velocity.

Here $V = 1850$ f. s.; $d = 8$ inches $= \frac{2}{3}$ feet; and $n = 30$. We have, therefore,

$$\text{Revolutions per second} = \frac{3 \times 1850}{2 \times 30} = 92.5;$$

$$\text{Velocity of surface} = \frac{3.1416 \times 1850}{30} = 193.7 \text{ f. s.};$$

$$\text{Angular velocity} \quad = \frac{2}{d} \times 193.7 = 581.1 \text{ f. s.}$$

Example 7. Make the same computations for a projectile fired from the 3.2-inch B. L. steel gun.

Here $V = 1608$ f. s.; $d = 3.2$ inches $= \frac{4}{15}$ feet; and $n = 30$. As before,

$$\text{R. per S.} = \frac{15 \times 1608}{4 \times 30} = 201;$$

$$\text{V. of S.} = \frac{3.1416 \times 1608}{30} = 168.4 \text{ f. s};$$

$$\omega = \tfrac{15}{2} \times 168.4 = 1263 \text{ f. s.}$$

Example 7. Make the same computations for the 30.5-cm. German (Krupp) gun.

Here $V = 523.6$ m. s., $d = 0.305$ m., and $n = 45$. We have

$$\text{R. per S.} = \frac{523.6}{45 \times 0.305} = 38.2;$$

$$\text{V. of S.} = \frac{3.1416 \times 523.6}{45} = 36.5 \text{ m. s.};$$

$$\omega = \frac{2}{0.305} \times 36.5 = 239.3 \text{ m. s.}$$

If the twist is given in degrees, we have

$$\text{R. per S.} = \frac{V}{\pi d} \tan \beta;$$

$$\text{V. of S.} = V \tan \beta;$$

$$\omega = \frac{2V}{d} \tan \beta.$$

Example 8. The French 27-cm. gun, pattern of 1875, has a twist of 4°, and gives to the projectile a muzzle velocity of 529 m. s.

Here $V = 529$, $d = 0.27$, and $\beta = 4°$.

$$\begin{aligned} \log V &= 2.7235 \\ \log \tan \beta &= 8.8446 \\ \text{a. c. } \log \pi &= 9.5029 \\ \text{a. c. } \log d &= 0.5686 \\ \hline & 1.6396 \qquad \therefore \text{ R. per S.} = 43.6 \end{aligned}$$

$$\log V \tan \beta = 1.5681 \qquad \therefore \text{ V. of S.} = 37.0 \text{ m. s.}$$

$$\begin{aligned} \log \frac{V \tan \beta}{d} &= 2.1367 \\ \log 2 &= 0.3010 \\ \hline & 2.4377 \qquad \therefore \omega = 274.0. \text{ m. s.} \end{aligned}$$

Moment of Inertia and Radius of Gyration.—The moment of inertia and radius of gyration of an oblong projectile are important factors in the drift formula. They may be determined experimentally by the principles of the compound pendulum, as explained in works on Mechanics.* They may also be computed with great accuracy for modern shot, which are carefully made of homogeneous material, symmetrically disposed about the axis of figure.

Moment of Inertia of an Ogival Head.—An ogival head is one half the solid of revolution generated by revolving a segment of a circle about its chord. If we take the geometrical axis of the ogive as the axis of x, and the origin in the plane of the base, we shall have for the equation of the generating curve,

$$y = \sqrt{4n^2R^2 - x^2} - (2n - 1)R.$$

* See Michie's Elements of Analytical Mechanics, second edition, page 167.

In this equation n is the number of *calibres*, or diameters of the projectile, in the radius of the circle; and R the radius of the base of the ogive. If we make $y = 0$, the resulting value of x is the length of the head. We have, therefore,

$$\text{Length of head} = R\sqrt{4n - 1}.$$

Let I_1 be the moment of inertia of an ogival head revolving about its geometrical axis; and k_1 the radius of gyration. Also let D be the density of the head, which we will suppose constant. Then it may be shown that

$$I_1 = \frac{\pi D}{2}\int_0^{R\sqrt{4n-1}} y^4 dx.$$

Substituting for y its value from the equation of the generating curve, integrating between the indicated limits and reducing, we have the following expression for the moment of inertia of an ogival head:

$$I_1 = \pi D R^5 F(n),$$

in which

$$F(n) = \frac{\sqrt{4n-1}}{30}\{840n^4 - 760n^3 + 238n^2 - 24n + 3\}$$
$$- 4n^2(2n-1)(7n^2 - 4n + 1)\sin^{-1}\frac{\sqrt{4n-1}}{2n}.$$

Let w_1 be the weight, in pounds, of a cubic foot of the material of which the head is made, and S its specific gravity. Then we shall have, since a cubic foot of water weighs 62.3687 pounds,

$$w_1 = 62.3687S,$$

and

$$D = \frac{w_1}{g}.$$

Making these substitutions, we have

$$I_1 = \frac{\pi w_1}{g} R^5 F(n).$$

In using this equation R must be expressed in feet.

Radius of Gyration of an Ogival Head.—To deduce an expression for the radius of gyration we have, by definition,

$$k_1^2 = \frac{I_1}{m_1}.$$

But

$$m_1 = \frac{w_1}{g} \times \text{vol.}$$

It may be shown by the Integral Calculus that the volume of an ogival head is expressed by the equation

$$\text{Vol.} = \pi R^2 F_1(n),$$

in which

$$F_1(n) = \tfrac{1}{8}(12n^2 - 4n + 1)(4n - 1)^{\frac{1}{2}} - 4(2n - 1)n^2 \sin^{-1}\frac{\sqrt{4n - 1}}{2n}.$$

Therefore, by substitution,

$$k_1^2 = R^2 \frac{F(n)}{F_1(n)};$$

or, if the radius be taken as the unit,

$$k_1 = \left\{ \frac{F(n)}{F_1(n)} \right\}^{\frac{1}{2}},*$$

which gives the radius of gyration in terms of the radius of the projectile.

The following table gives the values of the functions of n most likely to be useful:

n	$F(n)$	$F_1(n)$	$\left(\frac{F(n)}{F_1(n)}\right)^{\frac{1}{2}}$
0.5	0.2667	0.6667	0.632
1.0	0.3921	1.0074	0.624
1.5	0.4862	1.2586	0.622
2.0	0.5647	1.4674	0.621
2.5	0.6336	1.6499	0.620
3.0	0.6937	1.8141	0.619

* These formulas for the moment of inertia and radius of gyration of an ogival head are believed to be new.

It will be seen from the last column that the radii of gyration of all ogival heads used in gunnery are practically the same; that is, about 0.62 of the radius of the projectile.

If n be made infinite, we have

$$\frac{F(n)}{F_1(n)} = \frac{8}{21};$$

and, therefore, for an ogive whose length is infinite, we have

$$k_1 = \sqrt{\tfrac{8}{21}} = 0.61721,$$

which gives the inferior limit of the numbers in the last column.

Moment of Inertia of the Cylindrical Part of a Projectile.—The cylindrical part of a projectile is a solid generated by the revolution of a rectangle about one of its sides, the axis of revolution being the axis of the projectile. Designate the moment of inertia of the cylinder by I_2 and its length, in calibres, by a. Then since the equation to the line generating the surface of the cylinder is

$$y = R,$$

we have

$$I_2 = \frac{\pi D}{2}\int_0^{2aR} R^4 dx = \pi D a R^5.$$

But

$$D = \frac{w_1}{g},$$

and therefore

$$I_2 = \frac{\pi w_1}{g} a R^5.$$

If L be the total length of a projectile, in calibres, we have, for the length of the cylindrical part,

$$a = L - \tfrac{1}{2}\sqrt{4n - 1}.$$

Radius of Gyration of Body of Projectile.—To determine the radius of gyration of the cylindrical part, or *body*, of a projectile, we have, from definition,

$$k_2^2 = \frac{I_2}{m_2}.$$

But

$$m_2 = \frac{w_1}{g}\,\text{Vol.} = \frac{\pi w_1}{g}\,2aR^2. \qquad \therefore\ k_2 = R\sqrt{\tfrac{1}{2}}.$$

Radius of Gyration of a Cored Shot or Shell.—Let I_3 be the moment of inertia, k_3 the radius of gyration, and m_3 the mass of the solid of revolution taken out from the interior of the shot to form the core. Also, let I, k and m refer to the entire shot. Then we shall have

$$k^2 = \frac{I_1 + I_2 - I_3}{m_1 + m_2 - m_3};$$

and if the shot be solid,

$$k^2 = \frac{I_1 + I_2}{m_1 + m_2}.$$

Example 9. Compute the radius of gyration of the 10-inch service cored projectile. For description, see Report of the Chief of Ordnance for 1885, page 427, and accompanying plate. We have,

Diameter of base of head, ...	9.97 in.
Mean diameter of body, ...	9.938 "
Length of head, ...	13.19 "
Radius of ogive, ...	2 calibres
Length of body, ...	16.99 in.
Length of projectile, ...	30.18 "

The core is made up first of a hemisphere whose diameter is 4.375 inches; next of a cylinder 1.95 inches long and 4.375 inches in diameter, then of a frustum of an ogive whose radius is 7.7714 calibres of the body of the core; and lastly of a hemi-

sphere whose diameter is 1.5 inches. We will suppose the ogive to be complete, and omit the small hemisphere—a supposition which will not sensibly affect the result.

The specific gravity, a mean of seventeen specimens, one from each lot cast, was 7.2549. From these data we find

$$w_1 = 452.48 \text{ pounds};$$

and, therefore, taking g at 32.16,

$$\frac{\pi w_1}{g} = 44.201.$$

For the head we have

$$R = \frac{9.97}{24} \text{ feet, and } n = 2;$$

$$\therefore I_1 = \frac{\pi w_1}{g} R^5 F(2) = 0.3088,$$

and

$$m_1 = \frac{\pi w_1}{g} R^3 F_1(2) = 4.6496.$$

For the body we have

$$R = \frac{9.938}{24} \quad \text{and} \quad a = \frac{16.99}{9.938};$$

$$\therefore I_2 = \frac{\pi w_1}{g} a R^5 = 0.9199,$$

and

$$m_2 = \frac{\pi w_1}{g} 2aR^3 = 10.7305.$$

For the core we have

$$R = \frac{4.375}{24} \text{ feet,}$$

$$n = \tfrac{1}{2} \text{ for the hemisphere and } 7.7714 \text{ for the ogive,}$$

and

$$a = \frac{4.375}{1.95};$$

$$\therefore\ I_3 = \frac{\pi w_1}{g} R^5 \{F(\tfrac{1}{2}) + a + F(7.7714)\} = 0.0164,$$

and

$$m_3 = \frac{\pi w_1}{g} R^3 \{F_1(\tfrac{1}{2}) + 2a + F_1(7.7714)\} = 1.2078.$$

Therefore, for the whole projectile we have

$$k = \left\{ \frac{0.3088 + 0.9199 - 0.0164}{4.6496 + 10.7305 - 1.2078} \right\}^{\frac{1}{2}} = 0.29247;$$

$$\therefore\ \frac{k^2}{R^2} = \left(\frac{0.29247}{0.41408}\right)^2 = 0.499.$$

Therefore, for cored shot similar to our cast-iron 10-in. shot, we have, very nearly,

$$\mu = 0.5.$$

For a solid 10-in. shot we have, by using the numbers given above,

$$k = \left\{ \frac{0.3088 + 0.9199}{4.6496 + 10.7305} \right\}^{\frac{1}{2}} = 0.28265;$$

and, therefore,

$$\mu = \left\{ \frac{0.28265}{0.41408} \right\}^2 = 0.466.$$

If in the expression for k^2 for solid shot, given on page 196, we substitute the expressions for the moments of inertia and masses already given, and reduce, we get

$$k^2 = R^2 \left\{ \frac{F(n) + a}{F_1(n) + 2a} \right\},$$

and therefore

$$\mu = \frac{k^2}{R^2} = \frac{F(n) + a}{F_1(n) + 2a}.$$

Therefore, since n and a are constant for similar projectiles (Prob. XVIII), μ must also be constant. Therefore, for all solid shot having ogival heads struck with radii of two calibres, and which are three calibres long, we have

$$\mu = 0.466.$$

For *cored shot* and *shell* μ is not constant, but varies with the calibre, and with the dimensions of the core.

For the common shell used with the French sea-coast guns we have *

$$\mu = 0.64,$$

and this value of μ we will adopt for our own shells.

Weight of Cored Shot.—The weight of the 10-in. cored shot can easily be found from the above calculations. We have found the mass to be 14.1723; and therefore

$$\text{Weight} = 32.16 \times 14.723 = 455.8 \text{ pounds.}$$

Centre of Gravity of an Ogival Head.—The centre of gravity of an ogival head, supposed to be made of homogeneous material, is evidently on its axis of figure. If $\bar{x}$ be the distance from the centre of the base of the head to the centre of gravity, it may be shown that

$$\bar{x} = \frac{(8n - 1)R}{12F_1(n)}.$$

Example 10. What is the value of $\bar{x}$ for a hemisphere whose radius is R?

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* Baills, Traité de Balistique Rationnelle, page 275. In Baills' notation we have $\mu = 4B^2$. On page 278 he gives for a solid shot, $\mu = 0.44$; which is a little less than the value we have found above, because the French shot is but $2\frac{1}{2}$ calibres long.

In this case we have $n = \frac{1}{2}$ and $F_1(n) = \frac{4}{5}$. We therefore readily find for a hemisphere,

$$\bar{x} = \tfrac{5}{8}R.$$

Example 11. Compute the value of $\bar{x}$ for the head of the 10-inch cored shot.

Here $n = 2$, $F_1(n) = 1.4679$, and $R = 4.985$ in., and we have

$$\bar{x} = \frac{15 \times 4.985}{12 \times 1.4679} = 4.245 \text{ inches.}$$

The centre of gravity of the body of a solid projectile is evidently its centre of volume; which is the middle point of its axis. To determine the centre of gravity of the 10-inch solid shot we proceed as follows: Let x be the distance from the middle point of the axis of the body to the centre of gravity of the projectile. Then we shall have, using numbers already found,

$$10.7305x = 4.6496(12.74 - x),$$

from which we find

$$x = 3.8515 \text{ inches.}$$

Therefore the distance of the centre of gravity of the projectile from the base is

$$8.495 + 3.851 = 12.346 \text{ inches.}$$

In the above examples no notice has been taken of the copper band and screw plug. These, however, would make but little difference in the results.

The centre of gravity of a cored shot or shell can be found by cutting from stiff, uniform card-board a profile of the projectile and core, and balancing it upon the fiducial edge of a ruler.

Total Muzzle Energy of an Oblong Projectile.—The total muzzle energy of a projectile is the sum of its energy of

translation and energy of rotation, at the muzzle; and is, therefore, expressed by the equation

$$E = \frac{w}{2g} V^2 + \frac{w}{2g}\left(k\omega\right)^2,$$

in which E is the total energy in foot-pounds.

But we have, when d is expressed in feet,

$$\omega = \frac{2\pi V}{nd} = \frac{\pi V}{nR};$$

$$\therefore\ (k\omega)^2 = \left(\frac{\pi V}{n}\right)^2 \frac{k^2}{R^2} = \mu\left(\frac{\pi V}{n}\right)^2.$$

Making these substitutions and factoring, we have

$$E = \frac{wV^2}{2g}\left\{1 + \left(\frac{\pi}{n}\right)^2 \mu\right\}.$$

It is evident from this last equation that if E_0 = energy of translation, and E_ω = energy of rotation, we shall have

$$E_0 = \frac{n^2}{\mu\pi^2} E_\omega.$$

Example 12. Suppose the muzzle velocity of the cored projectile of Ex. 9 to be 1850 f. s., and the twist of the rifling to be one turn in 40 calibres. What is the total muzzle energy of the projectile?

Applying known numbers and dividing by 2240 to reduce to foot-tons, we have

$$E = 10861 \text{ foot-tons},$$

and this represents the total effective work performed by the powder charge upon the projectile.

We also find

$$E_0 = 324.9 E_\omega;$$

that is, the energy of translation in this case is more than three hundred times that of rotation.

Problem XXI.

To determine the Probability of Fire, and the Precision of Fire-arms.

Preliminary Considerations.—Suppose we fire a considerable number of shots from the same gun at a rectangular target of a sufficient size to receive all the hits. It is clear that if all the shots were fired under precisely the same physical conditions, they would all describe the same trajectory, and strike the target at the same point. But this result does not occur in practice. Numberless causes, for the most part beyond our control, conduce to the scattering of the points of impact over the target; and only a relative efficiency is attained, no matter how skilfully the gun may be laid.

Among the causes producing inaccuracy of fire may be mentioned the following: eccentricity and variations in the form and weight of the projectiles; variations in the weight and quality of the powder used and in the density of loading, producing variations in the muzzle velocity; variations in the jump—in the density of the air and direction and velocity of the wind along the track of the projectile—in the atmospheric refraction; imperfect centring of the projectile as it emerges from the bore, etc.

In view of all these accidental and unavoidable causes of inaccuracy, it becomes important to be able to answer, at least approximately, the following questions:

In a well-directed fire in which all possible precautions have been taken to insure accuracy, what are the chances of hitting a target of known dimensions and distance from the gun? With a given battery, what is the best distance at which to engage the enemy? What is the relative efficiency of different guns—not only those already in service but also those which may be proposed for adoption? And other similar questions.

Centre of Impact.—Though the points of impact (hits) of a great number of shots fired in a uniform manner from the same gun, at a particular point of a vertical target, are scattered all over the target, it will be seen at once that those within a certain area are more densely crowded together, and that they are more or less symmetrically disposed about a certain point of this area. This point is called the *centre of fire* or *centre of impact*, and its position on the target is determined as follows:

We will, for simplicity, refer the points of impact to a system of rectangular co-ordinates; and, to avoid negative values, will choose for the origin the lowest, left-hand point of impact on the target, the axis of x being vertical.

Let $x_1, x_2 \ldots x_n$ be the abscissas of the n points of impact, and X_0 their arithmetical mean. That is, let

$$X_0 = \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{\Sigma x}{n} \ldots, \qquad (1)$$

indicating by Σ the sum of similar quantities. Through the point X_0 draw a horizontal line. Similarly, let $y_1, y_2 \ldots y_n$ be the ordinates of the n points of impact, and

$$Y_0 = \frac{\Sigma y}{n}.$$

Through Y_0 draw a vertical line. The intersection of these two lines determines the centre of impact, whose co-ordinates are therefore X_0 and Y_0.

Designate the perpendicular distances of the n points of impact from the horizontal line drawn through X_0 (which are called *vertical deviations*), by $a_1, a_2 \ldots a_n$; that is, make

$$a_1 = x_1 - X_0, \quad a_2 = x_2 - X_0 \ldots a_n = x_n - X_0.$$

Then by addition we have

$$\Sigma a = \Sigma x - nX_0$$

or

$$\frac{\Sigma a}{n} = \frac{\Sigma x}{n} - X_0,$$

and therefore by (1)

$$\Sigma a = 0.$$

Similarly, if the *horizontal deviations* are represented by $b_1, b_2 \ldots b_n$, we shall have

$$\Sigma b = 0.$$

That is, the algebraic sum of the vertical (or horizontal) deviations with reference to the centre of impact, is zero; and in a great number of shots their distribution upon the target with reference to a horizontal, or vertical, line passing through the centre of impact, will assume a considerable degree of regularity, and the number of hits in each of the four quadrants around this centre will be nearly equal, and the ratio of these numbers very nearly unity.

The Sum of the Squares of the Vertical (or Horizontal) Deviations with Reference to the Centre of Impact, is a Minimum.—That is, this sum is less than if the deviations were measured from any other point whatever.

Let X be the abscissa of any other point on the target, and $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ the *vertical deviations* with reference to this point. That is, let

$$\epsilon_1 = x_1 - X, \quad \epsilon_2 = x_2 - X \ldots \epsilon_n = x_n - X.$$

Squaring these n equations and adding the results, we have

$$\Sigma\epsilon^2 = \Sigma x^2 - 2X\Sigma x + nX^2.$$

Adding and subtracting nX_0^2 in the second member, and replacing Σx by its value from (1), we have

$$\Sigma\epsilon^2 = \Sigma x^2 - nX_0^2 + n(X - X_0)^2 \ldots \qquad (2)$$

The first two terms of the second member are constant with reference to ϵ, and therefore $\Sigma\epsilon^2$ varies only with $(X - X_0)^2$, which is essentially positive; and therefore $\Sigma\epsilon^2$ is a minimum when $X = X_0$.

We therefore have in this case, $\epsilon_1 = a_1$, $\epsilon_2 = a_2 \ldots$; and therefore from (2) we have, always,

$$\Sigma a^2 = \Sigma(x - X_0)^2 = \Sigma x^2 - nX_0^2.$$

In the same manner it may be shown that the sum of the squares of the vertical deviations is a minimum when $Y = Y_0$, and therefore

$$\Sigma b^2 = \Sigma(y - Y_0)^2 = \Sigma y^2 - nY_0^2.$$

By these last two formulas we can determine the sum of the squares of the deviations (which will be required further on) without determining the deviations themselves; and this is important when we have a considerable number of shots, and especially so when the co-ordinates X_0 and Y_0 are carried to a greater degree of approximation than the observed deviations.

Absolute Deviations.—Absolute deviations are the distances of the points of impact from the centre of impact. Designating these by $c_1, c_2 \ldots c_n$, we shall have $c_1 = \sqrt{a_1^2 + b_1^2}$, $c_2 = \sqrt{a_2^2 + b_2^2} \ldots$ Therefore the sum of the absolute deviations is

$$\Sigma c = \Sigma \sqrt{a^2 + b^2},$$

which is essentially positive.

We have already shown that Σa^2 and Σb^2 are minima, and therefore

$$\Sigma c^2 = \Sigma a^2 + \Sigma b_2$$

is also a minimum. That is, *the sum of the squares of the absolute deviations is a minimum.* This is what we should expect from the symmetrical grouping of the shots about the centre of impact.

Centre of Impact on a Horizontal Target.—Vertical targets being necessarily of moderate size, are employed at the shorter ranges only. They should be used whenever practicable, because by so doing errors due to inequalities of the ground are eliminated. At long ranges we employ the ground (or the

surface of the water if fired at sea) as a horizontal target. In this case we will take for the axis of y the trace upon the ground of the vertical target on which aim is taken; and for the axis of x the parallel to the plane of fire, drawn through the left lower corner of the target. The centre of impact of the shots upon the ground will be determined as before by the co-ordinates

$$X_0 = \frac{\Sigma x}{n} \quad \text{and} \quad Y_0 = \frac{\Sigma y}{n}.$$

The deviations, which are always referred to the centre of impact, are classified on a horizontal target, as lateral and longitudinal, the latter corresponding to the vertical deviations before considered. We may assume that the trajectory which passes through a point on the vertical target will also pass through the corresponding *point of fall* on the horizontal target; and also that the portion of the trajectory joining the two points is a straight line. If therefore x is the height of a particular shot on a vertical target, and x_1 the horizontal distance from the foot of the target to the point of fall, we shall have

$$x = x_1 \tan \omega,$$

in which ω is the angle of fall. By means of this formula we may transfer points from a horizontal to a vertical target, and *vice versa*. The lateral deviations will be practically the same for both targets.

Law of the Deviation of Projectiles.—The deviations of projectiles are analogous to the errors committed in the direct measurement of a magnitude of any kind. In fact, if we fire a great number of shots against a target under precisely similar circumstances, it will be found that the points of impact are the nearest together in the immediate vicinity of the centre of fire, and that these points lie more and more scattered the further we recede from this centre. The likelihood then of obtaining a particular deviation diminishes rapidly as the latter increases, a limit existing beyond which there will be no de-

viation. Errors of observation follow an entirely similar law; and therefore we can apply to both the same general principles.

We will confine ourselves at present to vertical deviations on a vertical target, since the formulas deduced will apply equally to horizontal or longitudinal deviations, and thus a great deal of repetition will be avoided.

Let then, as before, $a_1, a_2 \ldots a_n$ be the n vertical deviations of a large number of shots, and make

$$\sqrt{\frac{\Sigma a^2}{n-1}} = E_x.$$

Then it may be shown by the calculus of probabilities that the probability P that the vertical deviation of an additional shot will not exceed a specified amount s is given by the integral

$$P = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt = F(t), \text{ say,}$$

in which

$$t = \frac{s}{E_x \sqrt{2}}.$$

Now suppose we fire a new series of shots under precisely the same conditions as those by means of which E_x was determined. Then the probability of any shot of the series not having a greater vertical deviation than $\pm s$ will be given by the formula

$$P = F(t);$$

and therefore the probable number of shots which may be expected to fall between the two horizontal parallel lines drawn, the one at a distance s above the centre of impact, and the other at the same distance below it, will be found by multiplying the number of shots by the fraction expressing the probability.

Mean Quadratic Deviations.—E_x is called the *mean vertical quadratic* deviation, and is computed from the vertical

deviations by the equation given above. We may also compute E_x directly from the abscissas of the n points of impact, as follows:

We have

$$\Sigma a^2 = \Sigma x^2 - nX_0^2.$$

But, from (1),

$$nX_0^2 = \frac{(\Sigma x)^2}{n};$$

whence, substituting in the equation defining E_x, we have

$$E_x = \sqrt{\frac{\Sigma a^2}{n-1}} = \sqrt{\frac{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}{n-1}}.$$

Similarly, if E_y is the *mean horizontal quadratic deviation*, we shall have

$$E_y = \sqrt{\frac{\Sigma b^2}{n-1}} = \sqrt{\frac{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}{n-1}}.$$

Mean Deviations.—We may also employ the *mean vertical* and *horizontal deviations* for finding the *probable deviation P;* and when the number of shots is very great, the labor will thus be considerably abridged. Let, then, e_x and e_y be the mean vertical and horizontal deviations, respectively; that is, let

$$e_x = \frac{\Sigma a}{n} \quad \text{and} \quad e_y = \frac{\Sigma b}{n},$$

in which the deviations are all taken with the positive sign. Then we have, for the probability of a deviation s,

$$P = F(t),$$

as before; while t is given by the equation

$$t = \frac{s}{e\sqrt{\pi}}.$$

As the sums of the positive and negative deviations in each direction are equal in absolute value, the mean deviations can be more easily obtained by dividing the sum of the positive deviations by half the number of shots. Let x_s be the abscissa of any point of impact greater than X_0; then the corresponding deviation will be expressed by $x_s - X_0$, and their sum, supposing them m in number, will have the value

$$\Sigma x_s - mX_0.$$

The mean horizontal deviation will therefore be

$$e_x = \frac{\Sigma x_s - mX_0}{\frac{1}{2}n}.$$

Similarly,

$$e_y = \frac{\Sigma y_s - mY_0}{\frac{1}{2}n}.$$

Relation between the Mean Quadratic and Mean Deviations.—When the number of shots is very great, the probabilities

$$P = F(t) = F\left(\frac{s}{E\sqrt{2}}\right)$$

and

$$P = F(t) = F\left(\frac{s}{e\sqrt{\pi}}\right)$$

are practically the same; and in the limit they would be identical. But for a small number of shots the quadratic deviation is more to be relied upon than the mean. In the limit we should have the relation

$$\frac{e}{E} = \frac{\sqrt{2}}{\sqrt{\pi}}.$$

From this we obtain the following relations between E and e:

$$E = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e = 1.253314e,$$

and

$$e = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} E = 0.797885E.$$

Expression for π in Terms of the Mean Quadratic and Mean Deviations.—We have from the above equation the following remarkable expression for the ratio of the circumference of a circle to its diameter, viz.,

$$\pi = 2\left(\frac{E}{e}\right)^2.$$

In these equations we have omitted the subscripts x and y, since the equations are general.

General Probability Table.—The table opposite gives the values of P with t as argument; and, also, the values of t with P as argument.

From this table we can take P when t is given, and, reciprocally, t when P is known. If the probability be given we take the corresponding value of t from the table, and then compute the probable deviation s, either vertical or horizontal, by the equation

$$s = \pm\, tE\sqrt{2}, \quad \text{or} \quad s = \pm\, te\sqrt{\pi},$$

according as we employ the quadratic or mean deviation.

If a certain deviation $\pm\, s$, vertical or horizontal, be given, we determine its probability by computing t by the equation

$$t = \frac{s}{E\sqrt{2}} \quad \text{or} \quad t = \frac{s}{e\sqrt{\pi}},$$

and take the probability corresponding to t from Table 1. This gives the probability that the deviation of a shot will not be greater than s in either direction from the centre of impact.

TABLE 1.

t	P	Diff.	t	P	Diff.	t	P	Diff.
0.00	.0000	226	0.84	.7651	111	1.68	0.9825	13
0.02	.0226	225	0.86	.7761	106	1.70	0.9838	12
0.04	.0451	225	0.88	.7867	102	1.72	0.9850	11
0.06	.0676	225	0.90	.7969	99	1.74	0.9861	11
0.08	.0901	224	0.92	.8068	95	1.76	0.9872	10
0.10	.1125	223	0.94	.8163	91	1.78	0.9882	9
0.12	.1348	221	0.96	.8254	88	1.80	0.9891	8
0.14	.1569	221	0.98	.8342	85	1.82	0.9899	8
0.16	.1790	219	1.00	.8427	81	1.84	0.9907	8
0.18	.2009	218	1.02	.8508	78	1.86	0.9915	7
0.20	.2227	216	1.04	.8586	75	1.88	0.9922	6
0.22	.2443	214	1.06	.8661	72	1.90	0.9928	6
0.24	.2657	212	1.08	.8733	69	1.92	0.9934	5
0.26	.2869	210	1.10	.8802	66	1.94	0.9939	5
0.28	.3079	207	1.12	.8868	63	1.96	0.9944	5
0.30	.3286	205	1.14	.8931	60	1.98	0.9949	4
0.32	.3491	203	1.16	.8991	57	2.00	0.9953	32
0.34	.3694	199	1.18	.9048	55	2.25	0.9985	11
0.36	.3893	197	1.20	.9103	52	2.50	0.9996	3
0.38	.4090	194	1.22	.9155	50	2.75	0.9999	1
0.40	.4284	191	1.24	.9205	47	∞	1.0000	
0.42	.4475	187	1.26	.9252	45	0.0443	0.050	
0.44	.4662	183	1.28	.9297	43	0.0888	0.100	
0.46	.4845	182	1.30	.9340	41	0.1337	0.150	
0.48	.5027	178	1.32	.9381	38	0.1791	0.200	
0.50	.5205	174	1.34	.9419	37	0.2253	0.250	
0.52	.5379	170	1.36	.9456	34	0.2724	0.300	
0.54	.5549	167	1.38	.9490	33	0.3208	0.350	
0.56	.5716	163	1.40	.9523	31	0.3708	0.400	
0.58	.5879	160	1.42	.9554	29	0.4227	0.450	
0.60	.6039	155	1.44	.9583	27	0.4769	0.500	
0.62	.6194	152	1.46	.9610	26	0.5342	0.550	
0.64	.6346	148	1.48	.9636	25	0.5951	0.600	
0.66	.6494	144	1.50	.9661	23	0.6608	0.650	
0.68	.6638	140	1.52	.9684	22	0.7329	0.700	
0.70	.6778	136	1.54	.9706	20	0.8134	0.750	
0.72	.6914	133	1.56	.9726	19	0.9062	0.800	
0.74	.7047	128	1.58	.9745	18	1.0179	0.850	
0.76	.7175	125	1.60	.9763	17	1.1631	0.900	
0.78	.7300	121	1.62	.9780	16	1.3859	0.950	
0.80	.7421	117	1.64	.9796	15	1.8214	0.990	
0.82	.7538	113	1.66	.9811	14	2.3268	0.999	

Probable Deviation.—When the probability is one-half, that is, when $P = \frac{1}{2}$, we find from Table I that $t = 0.4769$. Therefore, calling the value of s in this case r, we have

$$r = \pm 0.4769E\sqrt{2} = \pm 0.6745E;$$

or, in terms of the mean deviation,

$$r = \pm 0.4769e\sqrt{\pi} = \pm 0.8453e.$$

r is called the probable deviation, that is, the deviation with respect to which the probabilities of obtaining greater or less deviations are equal. In other words, if we have fired a considerable number of shots at a target, under the same conditions, we may expect that half the deviations will be less than the probable deviation and the other half greater; and, generally, the probability of obtaining a deviation less (or greater) in absolute value, than the probable deviation is one-half.

Fifty-per-cent Zones.—We may, therefore, expect to find one-half the points of impact on the target lying within a zone of indefinite length whose sides are horizontal right lines at distances from the centre of impact equal to $+r$ and $-r$, and whose breadth is therefore $2r$. This is called the fifty-per-cent horizontal zone, and its breadth will be denoted by Z_x. We therefore have

$$Z_x = 2 \times 0.6745E_x = 1.349E_x;$$

or, in terms of the mean deviation,

$$Z_x = 2 \times 0.8453e_x = 1.691e_x.$$

Similarly, we have for the breadth of the fifty-per-cent vertical zone,

$$Z_y = 1.349E_y;$$

or, in terms of the mean deviation,

$$Z_y = 1.691e_y.$$

Twenty-five-per-cent Rectangle.—The intersection of these two zones determines a definite rectangle whose centre is

the centre of impact, and whose sides are parallel to the co-ordinate axes, and which will probably contain twenty-five per cent (fifty per cent of fifty per cent) of all the shots. This rectangle is called the *twenty-five-per-cent rectangle.*

Probable Rectangle.—If we wish to determine a rectangle which shall probably contain fifty per cent of all the shots, we must determine the breadth of a horizontal and of a vertical zone, each having a probability equal to the square root of one-half; and, therefore, giving by their intersection a rectangle whose probability is $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}} = \frac{1}{2}$.

We have $\sqrt{\frac{1}{2}} = 0.7071 = P$; and from Table 1 we find the corresponding value of $t = 0.7438$. Therefore, in this case,

$$s = \pm\, 0.7438E\sqrt{2} = 1.0519E;$$

or, in terms of the mean deviation,

$$s = \pm\, 0.7438e\sqrt{\pi} = 1.3183e.$$

Multiplying by 2, and designating the side of a rectangle by S, we have the following expressions for the sides of the probable rectangle:

$$S_x = 2.104E_x = 2.637e_x;$$

$$S_y = 2.104E_y = 2.637e_y.$$

Example 1. From the record of firing with a M. L. rifled mortar, at Sandy Hook, April 20, 1886, we take the following data:

No. of round.	x (yards.)	y (yards.)	a	b
178	0	4	-93.11	-4.67
179	84	16	-9.11	$+7.33$
180	32	9	-61.11	$+0.33$
181	163	12	$+69.89$	$+3.33$
182	209	0	$+115.89$	-8.67
183	54	6	-39.11	-2.67
184	56	10	-37.11	$+1.33$
185	144	12	$+50.89$	$+3.33$
186	96	9	$+2.89$	$+0.33$

Mean range 3357 yards.

Adding the values of x and y, we find

$$\Sigma x = 838 \quad \text{and} \quad \Sigma y = 78\,;$$

$$\therefore X_0 = \tfrac{838}{9} = 93.11, \quad Y_0 = \tfrac{78}{9} = 8.67.$$

We next find Σa^2 and Σb^2 as follows:

$$\Sigma x^2 = 114334 \quad \text{and} \quad \Sigma y^2 = 858\,;$$

$$\therefore \Sigma a^2 = 114334 - \frac{(838)^2}{9} = 36306.89\,;$$

$$\Sigma b^2 = 838 - \frac{(78)^2}{9} = 182\,;$$

and finally, employing the quadratic deviations,

$$E_x = \left(\frac{36306.89}{8}\right)^{\frac{1}{2}} = 67.37\,;$$

$$E_y = (\tfrac{182}{8})^{\frac{1}{2}} = 4.77.$$

The breadths of the 50-per-cent zones are, therefore,

$$Z_x = 1.349 \times 67.37 = 90.88 \text{ yards}\,;$$

$$Z_y = 1.349 \times 4.77 = 6.43 \text{ yards}\,;$$

and these are the sides of the 25-per-cent rectangle. For the probable rectangle we have

$$S_x = 2.104 \times 67.37 = 141.75 \text{ yards}\,;$$

$$S_y = 2.104 \times 4.77 = 10.04 \text{ yards}.$$

The probable deviations are, of course, one-half the breadths of the 50-per-cent zones; or, in round numbers, 45 yards and 3 yards, respectively.

To ascertain the actual number of hits in these rectangles, for comparison, we make use of the last two columns of the above table, which give the values of a and b, or the co-ordinates

of the points of impact with reference to the centre of impact as origin. These are calculated by the formulas

$$a = x - X_0$$

and

$$b = y - Y_0.$$

We find the number of hits in the probable rectangle to be 3, and in the 50-per-cent rectangle 6. These numbers should be $\frac{9}{4}$ and $\frac{9}{2}$, respectively.

Example 2. Thirty shots were fired at Meppen, December 20, 1880, at a target 1000 m. distant, with a 12-cm. siege-gun, weight of projectile 16.5 kg., weight of charge 4.5 kg. of prismatic powder, giving a M. V. of 512 m. s. All the shots struck the target to the left of the line of fire, and all but three below the centre. The following table gives the co-ordinates of the points of impact with reference to a suitably chosen origin, in centimetres:

No.	x	y	No.	x	y	No.	x	y
1	35	75	11	35	55	21	60	20
2	50	40	12	65	65	22	0	20
3	80	55	13	65	70	23	0	25
4	40	25	14	65	45	24	25	95
5	150	75	15	55	25	25	70	65
6	145	25	16	70	80	26	0	75
7	110	0	17	35	55	27	60	70
8	75	10	18	75	25	28	50	60
9	80	20	19	55	40	29	0	50
10	35	35	20	40	105	30	50	35

Adding the values of x and y, we find $\Sigma x = 1675$ and $\Sigma y = 1440$.

$$\therefore X_0 = \frac{1675}{30} = 55.83 \text{ cm., and } Y_0 = \frac{1440}{30} = 48 \text{ cm.}$$

We next find Σa^2 and Σb^2 as follows:

$$\Sigma x^2 = 131625, \quad \Sigma y^2 = 88850;$$

$$\therefore \ \Sigma a^2 = 131625 - 30 \times (55.83)^2 = 38104.17,$$

$$\Sigma b^2 = 88850 - 30 \times (48)^2 = 19730,$$

and finally

$$E_x = \left(\frac{38104.17}{29}\right)^{\frac{1}{2}} = 36.248;$$

$$E_y = \left(\frac{19730}{29}\right)^{\frac{1}{2}} = 26.083.$$

The breadths of the 50-per-cent zones are

$$Z_x = 1.349 \times 36.248 = 48.9;$$

$$Z_y = 1.349 \times 26.083 = 35.2;$$

and these are the sides of the 25-per-cent rectangle.

For the sides of the probable rectangle we have

$$S_x = 2.104 \times 36.248 = 76.3;$$

$$S_y = 2.104 \times 26.083 = 54.9.$$

The actual number of points of impact in this case (determined as in Ex. 1), in the 25-per-cent rectangle, is 10; and in the probable rectangle, 16. These numbers are, by theory, $7\frac{1}{2}$ and 15, respectively.

Example 3. Fifty shots were fired with the gun of Ex. 1, December 17, 1880, at an angle of elevation of 5°, giving a mean range of 2894.3 m.

Taking for the axis of x a line drawn through the point of fall farthest to the left, and parallel to the plane of fire, and for the axis of y a line drawn perpendicular to the plane of fire through the point of fall giving the shortest range, we have the

following co-ordinates of the fifty points of fall upon the ground, in metres:

No.	x	y	No.	x	y	No.	x	y
1	69	5.0	18	44	2.8	35	14	2.0
2	69	2.5	19	44	2.5	36	14	1.5
3	61	4.7	20	45	3.0	37	9	1.0
4	56	3.5	21	47	1.3	38	10	1.0
5	55	2.7	22	34	3.5	39	9	2.0
6	56	1.5	23	33	3.8	40	6	0.7
7	54	3.8	24	34	3.1	41	6	1.0
8	35	5.0	25	31	4.0	42	3	2.5
9	35	3.1	26	25	2.5	43	4	3.0
10	36	3.0	27	26	2.5	44	2	3.5
11	38	3.0	28	27	2.5	45	2	3.8
12	38	4.5	29	26	4.2	46	1	4.5
13	39	4.5	30	24	3.5	47	9	1.5
14	39	5.5	31	24	4.0	48	22	4.0
15	40	1.1	32	23	3.5	49	21	3.5
16	41	1.7	33	23	4.9	50	0	0.0
17	41	2.4	34	19	5.5			

REMARK. *These shots are not given in the order they were fired.*

From these co-ordinates we obtain $\Sigma x = 1463$ and $\Sigma y = 150.1$. The co-ordinates of the centre of impact with reference to the assumed origin are therefore

$$X_0 = \frac{1463}{50} = 29.26 \text{ m.};$$

$$Y_0 = \frac{150.1}{50} = 3.002 \text{ m.}$$

For the mean deviations e_x and e_y we have $\Sigma x_s = 1114$, $\Sigma y_s = 98.9$, $m = 25$, for deviations in range, and $m = 24$, for lateral deviations. Therefore

$$e_x = \frac{1114 - 25 \times 29.26}{25} = 15.3 \text{ m.};$$

$$e_y = \frac{98.9 - 24 \times 3.002}{25} = 1.07408 \text{ m.}$$

We next compute $\Sigma x^2 = 59605$, and $\Sigma y^2 = 537.44$; whence

$$\Sigma a^2 = 59605 - \frac{(1463)^2}{50} = 16797.62;$$

$$\Sigma b^2 = 537.44 - \frac{(150.1)^2}{50} = 86.8398:$$

and therefore

$$E_x = \left(\frac{16797.62}{49}\right)^{\frac{1}{2}} = 18.515;$$

$$E_y = \left(\frac{86.8398}{49}\right)^{\frac{1}{2}} = 1.331.$$

We are now prepared to compute the sides of the 25-per-cent rectangle, the sides of the probable or any other proposed rectangle, and the probable deviations. We have, using the quadratic deviations,

$$\left.\begin{aligned} 1.349 \times 18.515 &= 25.0 \text{ m.} \\ 1.349 \times 1.331 &= 1.8 \text{ m.} \end{aligned}\right\} \text{Sides of 25-per-cent rectangle.}$$

$$\left.\begin{aligned} 2.104 \times 18.515 &= 39.0 \text{ m.} \\ 2.104 \times 1.331 &= 2.8 \text{ m.} \end{aligned}\right\} \text{Sides of probable rectangle.}$$

The probable deviations are, respectively, one-half the sides of the 25-per-cent rectangle. Therefore the probable longitudinal deviation, or probable deviation in range, is 12.5 m. The probable lateral deviation is 0.9 m.

The origin of co-ordinates is 22.5 m. to the left of the plane of fire. The mean lateral deviation *from the plane of fire* is therefore

$$22.5 - Y_0 = 22.5 - 3.002 = 19.498 \text{ m.}$$

to the left.

The origin is also 2865 m. from the gun. The mean range is therefore

$$2865 + X_0 = 2865 + 29.26 = 2894.26 \text{ metres.}$$

Comparing these results with the experiments (as in Ex. 1), we find 13 hits within the 25-per-cent rectangle, whereas by theory there should be 12.5; also 22 hits within the probable rectangle instead of 25. One-half the shots have a longitudinal deviation less than the probable deviation, and the other half greater; while of the lateral deviations, 24 are less and 26 greater than the probable deviations.

If we employ the mean deviations instead of the mean quadratic deviations, we have

$$\left.\begin{array}{l} 1.691 \times 15.3 \quad = 25.9 \text{ m.} \\ 1.691 \times \ 1.014 = 1.8 \text{ m.} \end{array}\right\} \text{Sides of 25-per-cent rectangle.}$$

$$\left.\begin{array}{l} 2.647 \times 15.3 \quad = 40.3 \text{ m.} \\ 2.637 \times \ 1.074 = \ 2.8 \text{ m.} \end{array}\right\} \text{Sides of probable rectangle.}$$

When the number of shots is considerable as in this example, it makes but little difference in the results whether we employ the mean or quadratic deviations.

Table for Computing Sides of Rectangles having a given Probability.—The following table is useful in solving examples similar to the above; that is, when we wish to determine the sides of a rectangle about the centre of impact, which will probably contain a given per cent of hits.

The table is constructed precisely as has been already illus trated in the case of the *probable rectangle*. That is, we enter Table 1, with the square root of the given probability P as the argument, and take out the corresponding value of t. We then have

$$\frac{S}{E} = 2\sqrt{2}t = 2.8284t.$$

TABLE 2.

P	$\frac{S}{E}$	P	$\frac{S}{E}$	P	$\frac{S}{E}$
0.05	0.568	0.40	1.802	0.75	2 997
.10	0.815	.45	1.952	.80	3.237
.15	1.013	.50	2.104	.85	3.524
.20	1.187	.55	2.260	.90	3.898
.25	1.349	.60	2.425	.95	4.473
.30	1.503	.65	2.599	.99	5.613
.35	1.653	.70	2.788	.999	6.937

In this table P is the given probability, or "probability per cent" as it is frequently called; S is either side of the rectangle; and E the corresponding quadratic deviation. If, in place of the quadratic deviation, we prefer to use the mean deviation, we must substitute $1.2533e$ for E. If the *probable deviation* is given, we employ the relation

$$E = \frac{r}{0.4769\sqrt{2}} = 1.4826r.$$

Example 4. Compute the sides of the rectangle whose centre is the centre of impact, which will probably contain 75 per cent of the shots of Ex. 3.

We have in this case $P = 0.75$, corresponding to which we find, from Table 2,

$$\frac{S}{E} = 2.997.$$

Therefore

$$S_x = 2.997E_x = 2.997 \times 18.515 = 55.5 \text{ m.},$$

$$S_y = 2.997E_y = 2.997 \times 1.331 = 3.989 \text{ m.},$$

which are the sides required. The actual number of hits within this rectangle was 38,—agreeing with theory.

Enveloping Rectangle.—For a probability of 90 per cent we find, from Table 2,

$$S = 3.898E,$$

and this rectangle includes nearly all the hits of Ex. 3. The rectangle which includes all the hits of a given number of shots is called the *enveloping rectangle of the shots*. We may say, as a general rule, that the sides of the enveloping rectangle do not exceed, respectively, 4 times the mean quadratic deviations or 5 times the mean deviations. That is, we may reasonably expect that no deviation will exceed these limits. Any shot falling outside the enveloping rectangle must be regarded as abnormal.

Comparison of Experiment with Theory.—For purposes of comparison we have computed the following table with the data of Ex. 3, showing the agreement between theory and practice. The computations are similar to those given in the solution of Ex. 4. The table explains itself.

Given probability.	Sides of Rectangle in Metres.		No. of Hits.	
P	S_x	S_y	Theory.	Observed.
.05	10.52	0.76	2.5	1
.10	15.09	1.08	5	9
.15	18.76	1.35	7.5	11
.20	21.98	1.58	10	11
.25	24.98	1.80	12.5	13
.30	27.83	2.00	15	16
.35	30.60	2.20	17.5	19
.40	33.36	2.40	20	21
.45	36.14	2.60	22.5	21
.50	38.96	2.80	25	22
.55	41.84	3.01	27 5	27
.60	44.90	3.23	30	28
.65	48.12	3.46	32 5	28
.70	51.62	3.71	35	31
.75	55.49	3.99	37.5	39
.80	59.93	4.31	40	43
.85	65.25	4.69	42.5	45
.90	72.17	5.19	45	47
.95	82.82	5.95	47.5	49

This table gives a tolerably clear idea of the confidence that may be placed in the solutions of problems having reference to probability of fire. By comparing the last two columns it will be seen that the agreement between theory and practice, though near enough for all practical purposes, is not exact in a single instance. For example, in the probable rectangle where, by theory, there should be 25 hits, we find but 22. But it may be shown by the calculus of probabilities that the probability of there being exactly 25 hits in the probable rectangle is quite small.

To show this, we will make use of Bernoulli's theorem of compound probabilities. If we designate by p the probability of a success, and by $q = 1 - p$ that of a failure, the probability

that in $m+n$ trials there may be m successes and n failures is expressed by the equation

$$P=\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot \cdot \cdot \cdot (m+n)}{(1 \cdot 2 \cdot 3 \ldots m)(1 \cdot 2 \cdot 3 \ldots n)}p^m q^n.$$

In our example the number of trials is 50; and we wish to determine the probability that 25 of them shall be successes and 25 failures—that is, that out of the 50 shots 25 shall strike within the probable rectangle (whose probability is $\frac{1}{2}$), and 25 without it. We, therefore, have

$$m=n=25;\ p=\tfrac{1}{2};\ q=1-\tfrac{1}{2}=\tfrac{1}{2};$$

and by substituting, the above expression becomes

$$P=\frac{1 \cdot 2 \cdot 3 \ldots 50}{(1 \cdot 2 \cdot 3 \ldots 25)^2}\left(\frac{1}{2}\right)^{50}.$$

In the reduction of the second member of this equation we will make use of Sterling's formula which gives the approximate value of the continued product of the natural numbers from 1 to x. This formula is

$$1 \cdot 2 \cdot 3 \cdot 4 \ldots x = e^{-x}x^x \sqrt{2\pi x}.$$

Substituting this in the expression for P we have

$$P=\frac{e^{-50}50^{50}\sqrt{100\pi}}{e^{-50}25^{50}\times 50\pi}\left(\frac{1}{2}\right)^{50}.$$

By cancelling factors common to the numerator and denominator of this fraction, we easily obtain

$$P=\frac{1}{\sqrt{25\pi}} \doteq 0.11+.$$

We see, then, that the probability of exactly 25 out of the 50 shots falling within the probable rectangle is but $\frac{11}{100}$. That is, if we should fire 100 series, of 50 shots each, under

precisely the same conditions, we should have no reason to expect that exactly 25 shots would fall within the probable rectangle more than eleven times.

In this case, what we learn from the calculus of probabilities is that the probability that the probable rectangle contains 25 hits is greater than that of its containing 20 or any other number fixed upon. Moreover, it follows from Bernoulli's theorem that the probability that the probable rectangle contains $25 + n$ hits is the same as that it contains $25 - n$ hits.

We have a right, therefore, to infer that in several similar series of shots the number of projectiles falling within the probable rectangle will be very nearly one-half the shots fired.

Example 5. With the data of Ex. 3, compute the probability corresponding to a transverse zone extending 22.5 metres on both sides of the centre of impact. Also determine the breadth of the longitudinal zone which shall have the same probability as this transverse zone; and, lastly, the probability of the rectangle formed by the intersection of these zones.

We have here, $S_x = 2 \times 22.5 = 45$; $E_x = 18.515$; and since

$$\frac{S}{E} = 2\sqrt{2}t,$$

we have, using the given numbers,

$$t = \frac{45}{2\sqrt{2} \times 18.515} = 0.8593.$$

Entering Table 1 with this value of t, we find, by interpolation,

$$P = 0.7651 + \frac{193}{200} \times 0.0110 = 0.7757,$$

which is the required probability.

To find the breadth of the longitudinal zone having a probability equal to that of the transverse zone, we evidently

have from the above equation, since t is the same for both zones, the relation

$$\frac{S_x}{E_x} = \frac{S_y}{E_y},$$

and therefore

$$S_y = \frac{S_x}{E_x} E_y.$$

We have found $E_y = 1.331$; and therefore

$$S_y = \frac{45}{18.515} \times 1.331 = 3.235 \text{ metres.}$$

That is, the required longitudinal zone extends 1.6+ metres on both sides of the centre of impact.

The probability of the corresponding rectangle is

$$0.7757 \times 0.7757 = 0.6017.$$

This rectangle should, therefore, according to the theory, contain $0.6017 \times 50 = 30$ hits. The actual number was 27.

Table for Computing the Width of a Zone of given Probability.—The following table facilitates the solution of examples similar to the above. The argument is either $\frac{S}{E}$ or $\frac{S}{e}$, according as the quadratic or mean deviations are employed.

S is the breadth of any zone either horizontal or vertical, transverse or longitudinal, whose centre coincides with the centre of impact. P_1 is the corresponding probability when quadratic deviations are employed; and P_2 the probability in terms of the mean deviations.

The table was computed as follows: For each value of the argument, $\frac{S}{E}$, the value of t was computed by the equation

$$t = \frac{S}{2\sqrt{2}E},$$

and the corresponding value of P taken from Table 1.

Similarly, for each value of $\frac{S}{e}$ the value of t was computed by the equation

$$t = \frac{S}{2\sqrt{\pi}\,e},$$

and then the value of P was taken from the table as before.

TABLE 3.

$\frac{S}{E}$ or $\frac{S}{e}$	P_1	Diff.	P_2	Diff.	$\frac{S}{E}$ or $\frac{S}{e}$	P_1	Diff.	P_2	Diff.
0.0	.0000	399	.0000	318	2.9	.8529	135	.7527	159
0.1	.0399	397	.0318	318	3.0	.8664	124	.7686	152
0.2	.0796	396	.0636	317	3.1	.8788	116	.7838	144
0.3	.1192	393	.0953	315	3.2	.8904	106	.7982	139
0.4	.1585	389	.1268	312	3.3	.9010	99	.8121	129
0.5	.1974	384	.1580	311	3.4	.9109	90	.8250	123
0.6	.2358	378	.1891	308	3.5	.9199	82	.8373	117
0.7	.2736	372	.2199	305	3.6	.9281	76	.8490	110
0.8	.3108	375	.2504	300	3.7	.9357	69	.8600	104
0.9	.3483	366	.2804	297	3.8	.9426	62	.8704	99
1.0	.3829	348	.3101	291	3.9	.9488	57	.8803	91
1.1	.4177	338	.3392	287	4.0	.9545	51	.8894	87
1.2	.4515	328	.3679	280	4.1	.9596	46	.8981	80
1.3	.4843	317	.3959	276	4.2	.9642	42	.9062	76
1.4	.5160	307	.4235	269	4.3	.9684	38	.9137	71
1.5	.5467	296	.4504	262	4.4	.9722	33	.9208	65
1.6	.5763	284	.4766	257	4.5	.9755	30	.9273	62
1.7	.6047	272	.5023	250	4.6	.9785	27	.9335	57
1.8	.6319	260	.5273	242	4.7	.9812	24	.9392	53
1.9	.6579	248	.5515	235	4.8	.9836	21	.9445	49
2.0	.6827	236	.5750	228	4.9	.9857	19	.9494	45
2.1	.7063	224	.5978	221	5.0	.9876	31	.9539	80
2.2	.7287	211	.6199	212	5.2	.9907	24	.9619	70
2.3	.7498	200	.6411	206	5.4	.9931	18	.9688	57
2.4	.7698	189	.6617	197	5.6	.9949	10	.9745	48
2.5	.7887	176	.6814	189	5.8	.9959	10	.9793	40
2.6	.8063	167	.7003	182	6.0	.9969	31	.9833	167
2.7	.8230	158	.7185	175	∞	1.0000		1.0000	
2.8	.8388	141	.7360	167					

Example 6. Employing the data of Ex. 3, required the probable number of shots that we may expect to fall within a rectangle whose centre is the centre of impact, and whose sides are $S_x = 30$ m., and $S_y = 1.5$ m.

We have for the transverse zone, employing the quadratic deviation,

$$\frac{S}{E} = \frac{30}{18.515} = 1.620.$$

With this argument we find from Table 3, by interpolation,

$$P_1 = 0.5763 + \frac{20}{100} \times 0.0284 = 0.5820,$$

which is the probability for the transverse zone.

Therefore the number of hits we may look for in this zone is $0.5820 \times 50 = 29$. The actual number found there is 27.

For the longitudinal zone we have

$$\frac{S}{E} = \frac{1.5}{1.331} = 1.127.$$

Therefore, for this zone,

$$P_1 = 0.4177 + \frac{27}{100} \times 0.0338 = 0.4268,$$

which is the probability for the longitudinal zone.

The probable number of shots in this zone is, therefore, $0.4268 \times 50 = 21$; which agrees with observation.

The probability of the rectangle of intersection of these two zones is the product of the separate probabilities.

$$\therefore\ P = 0.5820 \times 0.4628 = 0.2484,$$

which is the probability required.

The probable number of shots in this rectangle is, therefore, $0.2484 \times 50 = 12 +$, while the actual number is 14.

Example 7. Required the dimensions of a *vertical* target large enough to receive all the shots of Ex. 3. Also determine the sides of the probable rectangle, and its position on the vertical target, supposing the mean angle of fall (ω) to be $8^\circ\ 11'$.

We will suppose the target to be placed with its left lower corner at the origin of co-ordinates. The shot having the longest range strikes the ground 69 m. beyond the target, which must therefore be at least 69 tan 8° 11′ = 9.9 m. high to receive this shot. The greatest deviation from the axis of x is 6 m. The required dimensions are therefore 9.9 m. high and 6 m. broad.

The mean lateral deviation and mean lateral quadratic deviation remain the same for the vertical as for the horizontal target, together with all that has been deduced from them. While the longitudinal deviations must be multiplied by tan ω to reduce them to vertical deviations. We therefore have for the sides of the probable rectangle

$$S_x = 39.0 \tan 8^\circ\ 11' = 6.6 \text{ m.};$$

$$S_y = 2.8 \text{ m.}$$

In the same way may the sides of any other rectangle on a vertical target be determined from those already found for the horizontal target.

The co-ordinates of the centre of impact on the vertical target are

$$X_0 = 29.26 \tan 8^\circ\ 11' = 4.2 \text{ m.};$$

$$X_0 = 3.0 \text{ m.}$$

Example 8. For the gun of Ex. 3 we have found the mean error in range to be 15.3 m., and in direction (lateral) to be 1.074 m., for a range of 2894 metres. At this range what is the probability of hitting with a single shot a horizontal target 41 m. by 2 m., the longer side being parallel to the plane of fire, and its centre coinciding with the centre of impact?

We have $e_x = 15.3$, $e_y = 1.074$. Therefore for the transverse zone

$$\frac{S}{e} = \frac{41}{15.3} = 2.680.$$

With this argument we find from Table 3, by interpola tion,

$$P_2 = 0.7003 + \frac{8}{10} \times .0182 = 0.7149.$$

For the longitudinal zone we have

$$\frac{S}{e} = \frac{2}{1.074} = 1.862.$$

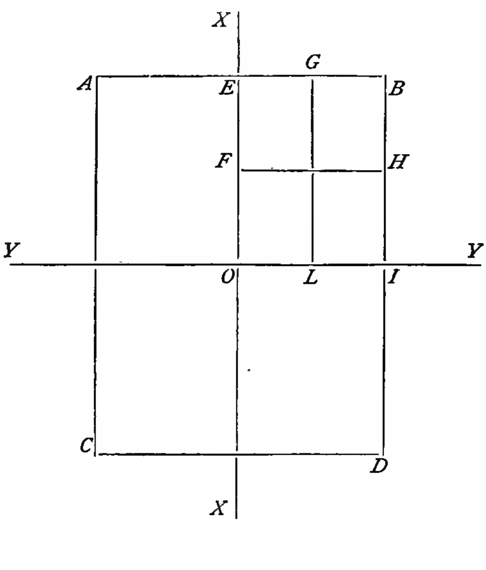
Therefore for this zone

$$P_2 = 0.5273 + \frac{62}{100} \times .0242 = 0.5423.$$

The probability of the rectangle of intersection is $0.7149 \times 0.5423 = 0.3877$; which is the probability required.

The probable number of shots in this rectangle is $0.3877 \times 50 = 19$; which agrees with observation.

Probability of Hitting any Plane Figure.—In what precedes we have given methods for computing the probability of hitting a rectangle whose sides are parallel to the co-ordinate axes, and whose centre coincides with the centre of impact. The same principles enable us to compute the probability of hitting, at least approximately, any plane figure.



In the diagram, suppose O to be the centre of impact of a number of shots upon a horizontal target, OX the direction in which longitudinal deviations, or deviations in range, are measured, and OY, perpendicular to OX, the direction in which lateral deviations are measured. In the last example we have shown that, with the data there used, the probability of hitting the rectangle $ABCD$ (not drawn to scale) is 0.3877, and that the number of shots actually falling within this rectangle agrees with the probability.

We may also assume, on account of the symmetrical grouping of the shots about the centre of impact O, that the probability of hitting the rectangle $OEBI$ is one-fourth that of hitting the rectangle $ABCD$.

The probability of hitting within the rectangle $OFHI$, or $OEGL$, is found in the same way as that of hitting within the rectangle $OEBI$. In the first case we should take $S_x = 20.5$ and $S_y = 2$ (see Ex. 8); and in the second case we should take $S_x = 41$ and $S_y = 1$.

The probability of hitting within $EBFH$ is found by subtracting the probability of the rectangle $FHOI$ from that of $OEBI$. In a similar manner we should find the probabilities of hitting within the rectangles $GBIL$, $EGKF$ and $IHKL$. Finally, the difference between the probabilities of the rectangles $GBIL$ and $IHKL$ is the probability of hitting within the rectangle $GBHK$.

In the same way we may divide up any plane figure into small rectangles, and the sum of their separate probabilities will be, approximately, the probability of hitting the figure.

Example 9. With the data of Ex. 8, what would be the probability of hitting the deck of a ship represented by a rectangle 80 m. by 16 m.—(*a*) when the ship is approaching the gun, bow on, and (*b*) when the ship is steaming perpendicular to the plane of fire?

We will suppose the centre of impact to be at the centre of the rectangle.

(*a*) We have for the transverse zone, $\frac{S}{e} = \frac{80}{15.3} = 5.229$; and for the longitudinal zone, $\frac{S}{e} = \frac{16}{1.074} = 15$.

The respective probabilities from Table 3 are 0.9639 and 1 (practically). The probability of hitting the deck is, therefore, for each shot, 0.9639.

(*b*) In this case we have for the transverse zone, $\frac{S}{e} = \frac{16}{15.3} = 1.046$; and for the longitudinal zone, $\frac{S}{e} = \frac{80}{1.074} = 74$. The probability of hitting the deck in this case is, therefore, 0.3235, or one-third what it was in the former case.

Example 10. If a zone of a certain breadth contains m per cent of a large number of shots fired, what is the breadth of another zone which will probably contain n per cent of the shots, supposing all the shots to be fired under precisely similar circumstances?

The given "per cents" are, for a large number of shots, the respective probabilities of a shot falling within the given zones. With these probabilities we enter Table 1 and take out the corresponding values of t. Then, since the deviations (mean or quadratic) are the same for both zones, it follows that the breadths of the zones are proportional to the values of t.

For example, for a 20-per-cent zone we find $t = 0.1791$; and for an 80-per cent zone, $t = 0.9062$. Therefore we have the proportion

$$0.1791 : 0.9062 :: 1 : 5.$$

That is, the 80-per-cent zone must be 5 times as wide as the 20-per-cent zone.

Example 11. If the 50-per-cent zones (horizontal and vertical) are each 6 feet wide for a certain gun and range, what is the probability of hitting a target 6 feet square, if the centre of impact is in the middle of the lower edge?

As the 50-per-cent vertical zone just includes the target, the probability for this zone will be 0.5.

We must next determine the probability of a horizontal zone double the height of the target, since the centre of impact is, by hypothesis, in the lower edge of the target. This is equivalent to determining the probability of a zone twice the breadth of the 50-per-cent zone. Then, since the breadths of zones for the same series of shots are proportional to the values of t, we have for the required zone

$$t = 2 \times 0.4769 = 0.9538.$$

With this argument, we find from Table 1 the probability of the zone to be 0.8226; which is the probability of a zone twice the breadth of the 50-per-cent zone. As the target is in the upper half of this zone, we divide the above probability by 2, which gives for the probability of the half-zone 0.4113. The probability for hitting the target is therefore

$$P = 0.5 \times 0.4113 = 0.2057.$$

If the centre of impact were in the centre of the target, the probability of hitting would evidently be

$$P = 0.5 \times 0.5 = 0.25.$$

If the centre of impact were raised 2 feet on the target, the probability of the vertical zone would still be 0.5. For the horizontal zones, we must take one 4 feet broad to get the probability of hitting that part of the target below the centre of impact; and one 8 feet broad to get the probability of hitting the upper part of the target. The values of t for these two zones are, respectively,

$$t = \tfrac{4}{6} \times 0.4769 = 0.3179,$$

and

$$t = \tfrac{8}{6} \times 0.4769 = 0.6359.$$

The corresponding zonal probabilities are, therefore, 0.3469 and 0.6315. One-half the sum of these probabilities is the probability of the horizontal zone which includes the target.

Finally, multiplying this last probability by the probability

of the vertical zone (0.5), we find the probability of hitting the target to be

$$P = 0.5 \times 0.4892 = 0.2446.$$

Example 12. There were fired at the Grüson turret, at Bucharest, on December 31, 1885, and January 1, 1886, 94 projectiles from a Krupp 21-cm. rifled mortar, planted at a distance of 2510 m. (2745 yards). The charge was 3 kg. of coarse-grained powder; weight of projectile, 91 kg.

The angles of elevation varied from 53° to 56° 30′; and the angles of fall from 57° 30′ to 61° 30′. Omitting shots numbered 61, 62 and 72, whose points of impact are not given in the work from which this data has been taken,* we find for the remaining 91 shots as follows: Mean deviation in range, 33.27 m. $= e_x$; and mean deviation in direction, 9.905 m. $= e_y$. The co-ordinates of the centre of impact with reference to the centre of the turret are, in range, $x = +0.277$ m., and in direction, $y = +0.275$ m. As the turret was 6 metres in diameter, it will be seen that the centre of impact of the 91 shots was within the circumference of the turret. This indicates very fine marksmanship; but as the turret was not hit, it is interesting to know the probability of hitting it.

Instead of the turret we will take its circumscribing square; and, further, will assume that the centre of impact coincides with the centre of the turret. We have

$$\frac{S_x}{e_x} = \frac{6}{33.27} = 0.1803.$$

The probability, therefore, of a shot striking within a transverse zone embracing the turret is, from Table 3, 0.0573.

We also have

$$\frac{S_y}{e_y} = \frac{6}{9.905} = 0.6057.$$

* Revue Militaire Belge, vol. 2, 1886, page 171.

Therefore the probability of a shot striking within a longitudinal zone including the target is, from Table 3, 0.1909.

The required probability is, therefore,

$$P = 0.0573 \times 0.1909 = 0.0109.$$

That is, in 100 shots we could not expect to hit the target more than once.

The dimensions of the 25-per-cent rectangle and probable rectangle are

$$\left.\begin{array}{l} 1.691 \times 33.27 = 56.26 \text{ m.} \\ 1.691 \times 9.905 = 16.75 \text{ m.} \end{array}\right\} \text{Sides of 25-per-cent rectangle.}$$

$$\left.\begin{array}{l} 2.637 \times 33.27 = 87.74 \text{ m.} \\ 2.637 \times 9.905 = 26.12 \text{ m.} \end{array}\right\} \text{Sides of probable rectangle.}$$

The enveloping rectangle, or the rectangle containing all the shots fired, was 200 m. (218.7 yards) long and 70 m. (76.55 yards) broad.

Curves of Equal Probability.—*A curve of equal probability is one for which the probability of projectiles striking its different points is constant.* In what has preceded it has been assumed that the perimeter of a rectangle enjoys this property; but a little consideration will show that this assumption, though perhaps accurate enough for most practical purposes in gunnery, is not strictly correct.

Curves of equal probability are ellipses (including the circle); and they enjoy the following characteristic properties, viz.: *Of all the equal plane areas that can be considered on the target (vertical or horizontal), that bounded by an ellipse of equal probability is that in which the probability of projectiles falling is the greatest.* From this cause the areas bounded by the ellipses mentioned have been called *areas of maximum probability.*

Viewing the question of probability of fire from a theoretical point of view, it appears natural to choose one of these ellipses as a criterion for forming a judgment of the precision of pieces; for example, that of 25 or 50 per cent of the shots. The area thus chosen can be considered to be made up of the different

elliptic rings, which include the elements to which the same probability attaches; to all the elements of which it is composed correspond greater probabilities than to those which are exterior to them, and the anomaly is not incurred which is met with in the 25- or 50-per-cent rectangles, where some shots are considered as acceptable whose probabilities are inferior to those of others which are rejected.*

Relations between the Semi-axes of Ellipses of Equal Probability and the Deviations.—It may be shown that the principal axes of ellipses of equal probability (which we will take parallel to the co-ordinate axes) are proportional to the respective deviations (mean, mean quadratic, or probable) in the same directions. That is, if a and b are the semi-axes, we have the relations, when the number of shots is very great,

$$\frac{a}{b} = \frac{r_x}{r_y} = \frac{E_x}{E_y} = \frac{e^x}{e^y}.$$

We also have

$$a = kE_x\sqrt{2} = ke_x\sqrt{\pi} = \frac{kr_x}{0.4769}$$

and

$$b = kE_y\sqrt{2} = ke_y\sqrt{\pi} = \frac{kr_y}{0.4769},$$

k being a constant whose value may be determined for any given value of a or b by one of the above equations.

Probability of a Projectile falling within an Ellipse of Equal Probability.—It may be shown that the probability P of a projectile falling within an ellipse of equal probability is given by the equation

$$P = 1 - e^{-k^2},$$

in which e is the Naperian base, and k a quantity defined by either one of the equations given in the preceding article.

* Translation of an article on the Precision of Fire-arms, published in the Memorial d'Artilleria. By Captain P. A. Macmahon, R. A. The author desires to express his great indebtedness to this admirable paper.

From this equation the probability of projectiles falling within a given ellipse can be easily computed.

Table for Computing the Semi-axes for a given Probability.—To determine the semi-axes of an ellipse corresponding to a given probability, we have from the above expression for P, substituting for k^2 its values in succession,

$$\frac{2a}{E_x} = \frac{2b}{E_y} = 2\sqrt{-2\log_e(1-P)}$$

and

$$\frac{2a}{e_x} = \frac{2b}{e_y} = 2\sqrt{-\pi\log_e(1-P)}.$$

By means of these formulas the following table was computed:

TABLE 4.

P	$\frac{2a}{E_x}$ or $\frac{2b}{E_y}$	$\frac{2a}{e_x}$ or $\frac{2b}{e_y}$	P	$\frac{2a}{E_x}$ or $\frac{2b}{E_y}$	$\frac{2a}{e_x}$ or $\frac{2b}{e_y}$
.05	0.641	0.803	0.55	2.527	3.168
.10	0.918	1.151	0.60	2.607	3.393
.15	1.140	1.429	0.65	2.898	3.632
.20	1.336	1.665	0.70	3.104	3.890
.25	1.517	1.901	0.75	3.330	4.174
.30	1.689	2.117	0.80	3.588	4.497
.35	1.856	2.327	0.85	3.896	4.883
.40	2.021	2.534	0.90	4.292	5.379
.45	2.187	2.741	0.95	4.895	6.136
.50	2.355	2.951	1.00	∞	∞

With the help of this table it is as easy to compute an *ellipse* corresponding to a given probability as to compute a *rectangle*. For example, for the *probable ellipse* we have $P = 0.50$; and therefore, from the table,

$$2a = 2.355E_x = 2.951e_x;$$

$$2b = 2.355E_y = 2.951e_y.$$

For the 25-*per-cent ellipse* we have

$$2a = 1.517E_x = 1.901e_x;$$
$$2b = 1.517E_y = 1.901e_y;$$

and similarly for any other ellipse.

Area of Probable Ellipse.—The area of the probable ellipse is $\pi ab = 4.3552E_xE_y = 6.8411e_xe_y$; while the area of the probable rectangle is (see page 213)

$$S_xS_y = 4.4268E_xE_y = 6.9538e_xe_y;$$

which is greater than the former, as has been already stated.

Equation of Probable Ellipse.—From the relations already given, it may easily be shown that the equation of an ellipse of equal probability is

$$y = \frac{E_y}{E_x}\sqrt{a^2 - x^2} = \frac{e_y}{e_x}\sqrt{a^2 - x^2} = \frac{r_y}{r_x}\sqrt{a^2 - x^2}.$$

In all the equations here given relating to ellipses of equal probability, the centre of the ellipse is supposed to coincide with the centre of impact, and the principal axes to be parallel to the axes X_0 and Y_0.

Example 13. Compute the probable ellipse and 25-per-cent ellipse with the data of Ex. 3.

We have $E_x = 18.515$ m., and $E_y = 1.331$ m.

Probable ellipse. $P = 0.50$

$$2a = 2.355 \times 18.515 = 43.6 \text{ m.};$$
$$2b = 2.355 \times 1.331 = 3.1 \text{ m.}$$

25-per-cent ellipse. $P = 0.25$

$$2a = 1.517 \times 18.515 = 28.1 \text{ m.};$$
$$2b = 1.517 \times 1.331 = 2.0 \text{ m.}$$

If we employ the mean deviations instead of the mean quadratic, we have $e_x = 15.3$ m. and $e_y = 1.074$.

Probable ellipse. $P = 0.50$

$2a = 2.951 \times 15.3 \;\;= 45.2$ m.;

$2b = 2.951 \times 1.074 = 3.2$ m.

25-per-cent ellipse. $P = 0.25$

$2a = 1.901 \times 15.3 \;\;= 29.1$ m.;

$2b = 1.901 \times 1.074 = 2.0$ m.

It will be seen that the maximum length and breadth ($2a$ and $2b$) of the ellipse of equal probability for any given probability are greater than the corresponding sides of the rectangle of the same probability. But as each point of the bounding ellipse has the same probability, it follows that in employing rectangles some hits are likely to be omitted which have a greater probability than others which are retained.

Table for Computing the Probability of a given Ellipse.—We have, for the probability required,

$$P = 1 - e^{-k^2}.$$

Therefore, employing mean quadratic deviations,

$$\log\,(1 - P) = -\,k^2 = -\,\frac{a^2}{2E_x^{\,2}} = -\,\frac{1}{8}\left(\frac{2a}{E_x}\right)^2.$$

Multiplying by the modulus of the common system of logarithms, we have

$$\log\,(1 - P) = -\,\frac{0.4342945}{8}\left(\frac{2a}{E_x}\right)^2.$$

Similarly,

$$\log\,(1 - P) = -\,\frac{0.4342945}{8}\left(\frac{2b}{E_y}\right)^2.$$

We also have, in terms of the mean deviations,

$$\log\,(1 - P) = -\,\frac{0.4342945}{4\pi}\left(\frac{2a}{e_x}\right)^2 = -\,\frac{0.4342945}{4\pi}\left(\frac{2b}{e_y}\right)^2.$$

The following table was computed by these formulas. The argument may be either $\frac{2a}{E_x}$ or $\frac{2b}{E_y}$, according as the length or breadth of the ellipse is given; and the required probability follows in the second column, under the heading P_1. We may also take for the argument either $\frac{2a}{e_x}$ or $\frac{2b}{e_y}$; and then the probability sought will be found in the fourth column headed P_2.

TABLE 5.

$\frac{2a}{E}$	P_1	Diff.	P_2	Diff.	$\frac{2a}{E}$	P_1	Diff.	P_2	Diff.
0.1	.0012	38	.0008	24	2.6	.5704	270	.4161	241
0.2	.0050	62	.0032	39	2.7	.5980	267	.4402	239
0.3	.0112	86	.0071	56	2.8	.6247	258	.4641	238
0.4	.0198	110	.0127	70	2.9	6505	248	.4879	235
0.5	.0308	132	.0197	85	3.0	.6753	239	.5114	231
0.6	.0440	154	.0282	100	3.1	.6992	228	.5345	228
0.7	.0594	175	.0382	115	3.2	.7220	217	.5573	223
0.8	.0769	194	.0497	127	3.3	.7437	206	.5796	218
0.9	.0963	212	.0624	141	3.4	.7643	194	.6014	213
1.0	.1175	229	.0765	153	3.5	.7837	184	.6227	208
1.1	.1404	243	.0918	165	3.6	.8021	173	.6435	201
1.2	.1647	257	.1083	175	3.7	.8194	161	.6636	195
1.3	.1904	269	.1258	186	3.8	.8355	151	.6831	188
1.4	.2173	279	.1444	195	3.9	.8506	141	.7019	182
1.5	.2452	287	.1639	204	4.0	.8647	130	.7201	175
1.6	.2739	293	.1843	211	4.1	.8777	120	.7376	167
1.7	.3032	298	.2054	219	4.2	.8897	112	.7543	161
1.8	.3330	302	.2273	224	4.3	.9009	102	.7704	154
1.9	.3632	303	.2497	229	4.4	.9111	93	.7858	146
2.0	.3935	303	.2726	234	4.5	.9204	86	.8004	139
2.1	.4238	301	.2960	237	4.6	.9290	78	.8143	133
2.2	.4539	299	.3197	239	4.7	.9368	71	.8276	125
2.3	.4838	294	.3436	241	4.8	.9439	64	.8401	119
2.4	.5132	290	.3677	242	4.9	.9503	58	.8520	112
2.5	.5422	282	.3919	242	5.0	.9561		.8632	

Example 14. With the data of Ex. 12, what is the probability of hitting the area bounded by an ellipse of equal probability whose length ($2a$) is 50 metres? Also calculate the breadth ($2b$) of the ellipse.

In this example we have, employing mean deviations,

$$2a = 50 \text{ m.}; \; e_x = 33.27 \text{ m.}; \text{ and } e_y = 9.905 \text{ m.}$$

Therefore

$$\frac{2a}{e_x} = \frac{50}{33.27} = 1.5029;$$

and by interpolation from Table 5 we get

$$P_2 = 0.1639 + \frac{29}{1000} \times 204 = 0.1645,$$

which is the probability required.

The value of $2b$ is found as follows:

We have the relation

$$\frac{2b}{e_y} = \frac{2a}{e_x} = \frac{50}{33.27}.$$

But $e_y = 9.905$.

$$\therefore\ 2b = \frac{50}{33.27} \times 9.905 = 14.886 \text{ metres.}$$

Example 15. What is the probability of hitting a vertical circle whose diameter is 16 inches, assuming the *mean deviation* in either direction to be 8 inches, and the centre of impact to coincide with the centre of the circle?

In this case we have $2a = 2b = 16$ inches, and $e_x = e_y = 8$ inches. We therefore have $\frac{2a}{e} = 2$; and from Table 5, $P_2 = 0.2726$. That is, a little more than one-fourth the shots would, in the long-run, strike the circle.

The assumption made above that the mean deviations (vertical and horizontal) upon a vertical target are equal is approximately correct in small-arm practice, especially by experts. In this case it will be evident by inspection of the target that the areas of equal density are approximately bounded by concentric circles whose centres are at the centre of impact. That is, the pencil of trajectories, approximately cylindrical and increasing in density toward its axis, is cut by

the plane of the target, nearly normally. Not so, however, when the bullets strike the ground. The ground, regarded as a horizontal target, cuts the pencil of trajectories at a very acute angle, causing the circles of equal probability on the vertical target to elongate into ellipses whose longer axes are parallel to the plane of fire and whose shorter axes practically remain the same.

If R is the radius of any circle of equal probability on a vertical target, and ω the mean angle of fall, or the angle with which the axis of the pencil of trajectories strikes the ground, then we shall have for the semi-major axis of the corresponding ellipse on the ground, approximately,

$$a = R \cot \omega.$$

Example 16. What is the probability of hitting an ellipse whose vertical axis is 20 inches and horizontal axis 16 inches, when the mean vertical deviation is 10 inches and mean horizontal deviation 8 inches?

In this case we have

$$\frac{2a}{e_x} = \frac{2b}{e_y} = 2.$$

The answer is therefore the same as in the last example.

Example 17. A marksman, at the end of the practice season, finds that just one-half of all the shots he has fired at the 200-yard range have struck the bull's-eye. What is his mean deviation at that range?

As one half of a large number of shots fired at the target have hit the bull's-eye, we may fairly assume that the probability of his hitting the bull's-eye is one-half; and that the centre of impact and centre of bull's-eye coincide.

Let R be the radius of the bull's-eye. Then we have, since $k^2 = \frac{R^2}{\pi e^2}$,

$$P = \tfrac{1}{2} = 1 - e^{-\frac{R^2}{\pi e^2}},$$

in which the e in the exponent must not be confounded with the Naperian base. From this equation we get

$$e = \frac{R}{\sqrt{\pi \log_e 2}}.$$

We may find the numerical value of e when R is given by means of Table 4. For example, suppose R to be 6 inches. Then, since $P = 0.5$, we have, from Table 4,

$$\frac{2R}{e} = \frac{12}{e} = 2.951.$$

$$\therefore\ e = \frac{12}{2.951} = 4.06 \text{ inches.}$$

That is, his mean deviation from the centre of the bull's-eye would be very nearly 4 inches. The same result would of course be obtained by working out the above formula.

Probability of Hitting a given Object. Supply of Ammunition.—If we wish to open a breach, demolish a bomb-proof, destroy an armored work, etc., we generally know, from experiments and calculations previously made and tabulated for use, the number of shots from guns similar to those at our command which must *strike* the work in order to accomplish the desired result. The very important question then arises, How many shots must be *fired* to *secure* the necessary number of *hits?* The answer to this question determines the amount of *ammunition* required and the *time* that must be allowed.

Let p be the probability of hitting the given surface, or object, with one shot. For example, if the object were the probable rectangle, the value of p would be $\frac{1}{2}$; if the 25-per-cent rectangle, p would be $\frac{1}{4}$; if the Grüson turret as in Ex. 12, p would be about 0.01; and so on. The value of p is determined by experiment, in advance, by methods already given.

Now, though we know that the probability of hitting, for example, the probable rectangle, or probable ellipse, is one-half, it by no means follows that one-half the shots will hit this

surface. The last two columns of the table on page 221 show that for the example there considered the actual number of hits was very approximately equal to pn; where p is the probability for each rectangle, and n the whole number of shots fired.

Therefore, if n_0 is the number of shots which must hit the object to insure the desired results, we may determine roughly the number of shots that must be fired, by the equation

$$n = \frac{n_0}{p}$$

We can find the probability that the number of hits will not vary in either direction from $n_0 = pn$ by more than a given number, y, in the following manner:

Let
$$h^2 = \frac{n}{2n_0(n - n_0)},$$
and
$$t = hy;$$

then the required probability is expressed by

$$P = F(t),$$

which may be taken from Table 1, with t as the argument.*

Example 18. Suppose 50 shots are fired from a gun under similar circumstances. What is the probability that the number of hits in the probable rectangle will not differ from 25 by more than 2?

We have $n = 50, p = \frac{1}{2}, n_0 = 25$, and $y = 2$. Therefore

$$h^2 = \frac{50}{50 \times 25}.$$

$$\therefore\ h = \tfrac{1}{5} \quad \text{and} \quad t = 2h = 0.4.$$

Therefore, from Table 1, we find

$$P = 0.43,$$

the probability required.

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* Les Projectiles, by Major Jouffret, chapter 7.

Example 19. What is the value of y in Ex. 18, when $P = 0.9$?

We find from Table 1 that, for $P = 0.9$, $t = 1.1631$. Therefore

$$y = \frac{t}{h} = 5.8.$$

There is, therefore, a probability of 0.9, or practical certainty, that out of 50 shots fired from the same gun, 19 *at least* will strike the probable rectangle.

Probability that at least one Shot will hit the Object.—To determine the probability P that *at least one* shot out of n shots fired shall strike the object, we use the formula

$$P = 1 - (1 - p)^n,$$

in which, as before, p is the probability of hitting with a single shot. Solving this with reference to n, we have

$$n = \frac{\log(1 - P)}{\log(1 - p)},$$

which gives the number of shots that must be fired to insure a probability P that the object will be hit *at least once*.

Example 20. What was the probability of hitting the Grüson turret at Bucharest (see Ex. 12) at least once in 100 shots? Here $n = 100$ and $p = 0.01$. Therefore

$$P = 1 - (0.99)^{100} = 0.63,$$

the probability required. There was, therefore, more than an even chance of hitting the turret with the shots fired.

Example 21. How many shots would have to be fired at the Grüson turret to secure a probability of $\frac{9}{10}$ that it would be hit at least once?

We have

$$n = \frac{\log \frac{1}{10}}{\log \frac{99}{100}} = \frac{\log 10}{\log 100 - \log 99} = 229.$$

As but 94 shots were fired at the turret it is not at all surprising that it was not hit.

Criterion for Rejecting Abnormal Shots.—In nearly all extended series of shots there will be found some which differ

so much from the others and from the mean as to indicate that there is something abnormal about them—though we may not be able to say exactly what it is ; and we therefore reject them in determining the mean deviation or dimensions of the probable rectangle.

In order not to leave this rejection to the arbitrary discretion of the computer, we give the following criterion, or rule, for rejecting abnormal shots. It was first given by Chauvenet, and has been taken from his Spherical and Practical Astronomy, Volume II, page 565.

We have seen (page 207) that if

$$t = \frac{s}{E\sqrt{2}} = \frac{s}{e\sqrt{\pi}},$$

then $F(t)$ (the values of which are given in Table I) multiplied by n, the number of shots, gives the probable number of deviations less than s, in the direction of either co-ordinate axis ; and hence the quantity

$$n - nF(t) = n\{1 - F(t)\}$$

expresses the number of deviations that may be expected to be greater than the limit s. But if this quantity is less than $\frac{1}{2}$, it will follow that a deviation of the magnitude s will have a greater probability against it than for it, and may therefore be rejected. The limit of rejection of a single doubtful shot according to this simple rule is, therefore, obtained from the equation

$$\tfrac{1}{2} = n\{1 - F(t)\},$$

or

$$F(t) = \frac{2n - 1}{2n}.$$

Having found $F(t)$ from this equation, we take the corresponding value of (t) from Table 1, and then compute the limiting value of s by either of the equations

$$s = Et\sqrt{2} = 1.4142Et,$$

$$s = et\sqrt{\pi} = 1.7725et.$$

If it be found that any deviation (a or b) is greater than s, the shot producing it should be rejected. Then with the remaining $n-1$ shots determine new values of E (or e) and s, and proceed as before—rejecting but one shot at a time.

Example 22. Determine the limit of rejection of one of the following shots fired at Meppen, February 21, 1882, with a 21-cm. mortar. Weight of charge, 4 kg.; weight of projectile, 91 kg.; angle of elevation, 30°; mean range, 3307.4 m.

No. of shot.	x (Métres.)	a (Métres.)
115	188	+49.6
116	0	−138.4
117	157	+18.6
118	171	+32.6
119	176	+37.6

Taking the point of fall of least range as the origin, the values of x are given in the second column, and the corresponding values of a in the third column. From these we find

$$e_x = 55.36 \text{ m.}$$

We also have

$$F(t) = \frac{2n - 1}{2n} = 0.9;$$

whence

$$t = 1.1631.$$

Therefore the limit of rejection is

$$s = 1.7725 \times 1.1631 \times 55.36 = 114.1 \text{ m.}$$

The shot numbered 116 must therefore be rejected. This reduces the mean error in range to 9 metres, instead of 55.36 m.

Probability of the Arithmetical Mean.—We have assumed (see page 203) that the values of the co-ordinates of the centre of impact, X_0 and Y_0, are, respectively, the arithmetical mean of the co-ordinates of the various points of impact; which is strictly true only at the limit—that is, when the number of

shots is supposed to be infinite. The arithmetical mean, however, gives the most plausible (if not the most probable) values of these co-ordinates, and is taken as the basis of all applications of the calculus of probabilities to the combination of direct measurements made upon a single quantity.

It is shown by writers on the method of least squares that *the probable error of the arithmetical mean is equal to the probable error of a single observation divided by the square root of the number of observations.** That is, if r_0 is the probable error of the arithmetical mean, we shall have

$$r_0 = \frac{r}{\sqrt{n}};$$

or, substituting for r its values given on page 212 we have

$$r_0 = \pm \frac{0.6745E}{\sqrt{n}} = \pm \frac{0.8453e}{\sqrt{n}}.$$

As an illustration of the above, it will be found that, in Ex. 1, the probable errors of the co-ordinates X_0 and Y_0 are, respectively, $\pm$ 15.15 yards and $\pm$ 1.07 yards. We may, therefore, write these co-ordinates as follows:

$$X_0 = 93.11 \pm 15.15 \text{ yds.};$$
$$Y_0 = 8.67 \pm 1.07 \text{ yds.}$$

The co-ordinates of the centre of impact in Ex. 2 may be written,

$$X_0 = 55.83 \pm 4.46 \text{ cm.};$$
$$Y_0 = 48.00 \pm 3.21 \text{ cm.}$$

Similarly those of Ex. 3 become

$$X_0 = 29.26 \pm 1.77 \text{ m.};$$
$$Y_0 = 3.00 \pm 0.13 \text{ m.}$$

* Elements of the Method of Least Squares, by Mansfield Merriman, Ph.D., page 147.

PROBLEM XXII.

To compute a range table.

A range table should be so constructed as to afford all the data necessary to enable the gun for which it was prepared to be properly and promptly layed in such a manner that its projectiles may hit a given object whose distance from the gun is known; and, also, to predict the probable effect of the shots upon the object. In its simplest form it consists of a series of computed trajectories pertaining to several different ranges taken as argument, disposed in regular order for ready reference, so that any range may be readily found in the table, and with it all the elements of the corresponding trajectory.

The constants upon which a range table is based relate chiefly to the projectile, and must be known in advance with all the precision possible. These are the calibre and weight of the projectile and the coefficient of reduction, from which are deduced the ballistic coefficient C for a standard density of the air; the muzzle velocity and the jump of the gun. With these data we can compute for a given range, by Prob. XII, the angle of departure, the angle of fall, the striking velocity, and the time of flight. These are the fundamental elements required in a range table; but it should also give the variations of the angles of departure due to variations of the muzzle velocity and of the density of the air, the drift of rifled projectiles, the danger-space, etc., all of which will be considered in their proper places.

Range Column.—The first column of a range table should contain the *ranges* for which the table is computed; and which, as already stated, constitute the argument of the table. The common difference of these ranges should be small enough to avoid unnecessary interpolations in taking out the angles of elevation. It would be better in most cases to extend the

table sufficiently to avoid all interpolations; that is, to make the common difference not greater than the probable error in the estimate of the range. For the longer ranges the differences could be taken greater than for the shorter ranges, which are presumably more accurately known than the former.

Angles of Departure.—In the second column of the table are placed the angles of departure corresponding to the ranges on the same line in the first column. In computing these from the formula

$$\sin 2\phi = AC,$$

proceed as follows: Write the logarithm of C upon a piece of paper, or, preferably, near the lower edge of a card, and place it successively above the logarithms of A found in the table of logarithms, writing down only the sums. This can easily be done and saves a great deal of labor. Next, find in the proper table the angles corresponding to these logarithmic sines, and write down the values of ϕ to the nearest minute, performing the division mentally. It will be necessary to correct the value of C from time to time for altitude, by the method of Prob. XIII.

Angles of Elevation.—It must be remembered that the angles of departure in the second column are not the angles of elevation employed in laying the gun, but are generally greater by the angle of jump. This latter can only be determined by experiment, and when found must generally be subtracted from the values of ϕ to give the angles of elevation. The jump is positive for all guns so far as is known, with the exception of the Hotchkiss rapid-firing guns, which are said to have a negative jump of about $5\frac{1}{2}$ minutes. For these guns the jump must be added to the angle of departure.

If the jump has been accurately determined by experiment, the column of "angles of elevation" may take the place of that of "angles of departure." But it is generally best to retain both columns.

Variations of the Angles of Departure or of Elevation.—Following the column of angles of elevation should be

columns of the variations of these angles due to variations of the density of the air or weight of projectile, and of the muzzle velocity.

We may deduce formulæ for these variations as follows, the details of which are omitted on account of their great length: Take the variations of equations (10) and (8) upon the supposition that X and V are constant, and reduce by means of equations (17), (18) and (20), and the values of the differentials of the S- and A-functions which are given in Appendix I. The result, all reductions having been made, is the following very simple expression for the variation of sin 2ϕ due to a variation of C:

$$\varDelta \sin 2\phi = -(B-A)\varDelta C;$$

or, taking the variation of the first member,

$$\varDelta\phi = -\frac{B-A}{2\cos 2\phi}\varDelta C.$$

If we make $\varDelta C = \pm \frac{C}{10}$, and multiply the second member by 3438 in order to reduce $\varDelta\phi$ to minutes of arc, we have finally

$$\varDelta\phi = \mp \frac{171.9C}{\cos 2\phi}\{B-A\}.$$

The upper sign is used when the variation of C is positive, and the lower sign when it is negative. This formula gives the variation of the angle of departure, and also of the angle of elevation, in minutes of arc, due to a variation of the ballistic coefficient of one-tenth of its value. This latter deviation may be due either to a variation of the density of the air, or of the weight of the projectile, or both. If of the former, it will appear as a variation of the factor $\frac{\delta_1}{\delta}$. The ballistic tables are constructed upon the supposition that this factor is unity; and, therefore, a variation of one-tenth of its value will make it either 0.9 or 1.1. A reference to Table III will show that these

limits are ample to cover all cases likely to occur in practice. For any other variation of $\frac{\delta_{\prime}}{\delta}$ we consider the variation of ϕ proportional to the variation of $\frac{\delta_{\prime}}{\delta}$. For example, if $\frac{\delta_{\prime}}{\delta}$ is 1.071 or 0.929, in either case the variation of ϕ will be numerically 71 one-hundredths of the variation given in the range table. The variations of the weights of modern projectiles designed for the same gun and service are so small that it will hardly ever be necessary to take account of them.

Variations of the Muzzle Velocity.—If the muzzle velocity is the only variable, we shall have for the corresponding variation of sin 2ϕ, deduced from eq. (18),

$$\Delta \sin 2\phi = C\Delta A;$$

in which ΔA depends upon ΔV. If we make $\Delta V = \pm 50$ f. s., the values of ΔA may be taken directly from the Δ_v column of Table **A**; and since these differences are negative, we have

$$\Delta \sin 2\phi = \mp C\Delta_v .$$

Taking the variation of the first member, and reducing to minutes of arc, we have

$$\Delta\phi = \mp \frac{1719}{\cos 2\phi} C\Delta_v .$$

This formula gives the variation of the angle of elevation due to a variation of $\pm$ 50 f. s. of the muzzle velocity. The upper sign is used when the velocity is increased, and the lower one when it is diminished.

The variations of the muzzle velocity are generally due to variations of the different elements of loading. These may be easily determined from Sarrau's formula for the velocity given on page 164. Taking the logarithms of both members and differentiating, we have

$$\frac{\Delta V}{V} = \frac{3}{8}\frac{\Delta\pi}{\pi} + \frac{1}{4}\frac{\Delta \cdot \Delta}{\Delta} + \frac{1}{8}\frac{\Delta d}{d} + \frac{3}{16}\frac{\Delta u}{u} - \frac{7}{16}\frac{\Delta w}{w}.$$

Of the variations in the second member of this equation we need consider, for the same gun, only the first two and last; and the first two may be united into one as follows: Since the density of loading is the ratio of the weight of the charge to the volume of the powder-chamber (which is constant), we have

$$\frac{\Delta . \Delta}{\Delta} = \frac{\Delta \pi}{\pi};$$

and, therefore,

$$\frac{3}{8}\frac{\Delta \pi}{\pi} + \frac{1}{4}\frac{\Delta . \Delta}{\Delta} = \frac{5}{8}\frac{\Delta \pi}{\pi}.$$

If, therefore, we suppose only the weight of charge or weight of projectile to vary, we have the two following expressions for the corresponding variations of the muzzle velocity:

$$\frac{\Delta V}{V} = \pm \frac{5}{8}\frac{\Delta \pi}{\pi};$$

$$\frac{\Delta V}{V} = \mp \frac{7}{16}\frac{\Delta w}{w}.$$

Time of Flight.—The "time of flight" column should come next. This is an important element, and should be given not only for time shells, but for solid shot and percussion-shells as well, to enable the gun to be properly aimed at a rapidly moving object. The methods of computing the time of flight are given and fully illustrated in Problems V and XII.

Drift.—The drift of the projectile should follow the time; and in connection therewith the number of points to be taken for its correction. The formula for computing the drift is given on page 180, and is followed by several illustrative examples.

Angle of Fall.—The next column of the range table should contain the angles of fall. These should be computed at first by the approximate formula

$$\sin 2\omega = BC,$$

and afterwards by the more rigid formula

$$\tan \omega = \frac{B}{A} \tan \omega.$$

The change of formulas should be made when their results differ by more than a minute of arc. The same labor-saving methods recommended for computing the angles of departure are applicable here; and, indeed, in nearly all the computations.

Striking Velocity and Penetration of Armor.—The last two columns give the striking velocity of the projectile and the thickness of wrought-iron armor it will penetrate. These are computed by the formula

$$v = u \frac{\cos \phi}{\cos \theta},$$

and Maitland's formula,

$$\tau = \frac{v}{608.3} \left(\frac{w}{d}\right)^{\frac{1}{2}} - 0.14d.$$

Instead of Maitland's formula for the penetration, we may employ the following:

$$\tau^{0.65} = \frac{v}{915.44} \frac{w^{\frac{1}{2}}}{d^{\frac{3}{4}}}.$$

This is the latest French, as Maitland's is the latest English, formula; and is considered applicable to guns ranging in calibre from 1.85 inches (47 mm.) to 10.63 inches (27 cm.). The English formula is the more conservative of the two, giving slightly less results; and is probably as correct as any of the numerous formulas that have been proposed.

Example 1. Compute a partial range table for the 8-inch B. L. rifle designed for the cruiser Baltimore, assuming the following data: $V = 2100$ f. s.; $d = 8$ inches; $w = 250$ pounds; $c = 0.9$, and therefore $\log C = 0.63752$.

The table is given on the next page. It extends from $X = 500$ yards to $X = 9000$ yards, with a common difference

of 500 yards. For practical use this common difference should be reduced to 100 yards (or even less) for reasons already given, reducing the elements to correspond by interpolation. Various other columns might be added to the table, such as the danger-spaces for different objects. Also, when the mean deviations shall have been determined by experiment, the dimensions of the probable rectangles, both horizontal and vertical, for the various ranges should be added.

RANGE TABLE FOR 8-INCH B. L. NAVAL GUN FOR SHELLS WEIGHING 250 LBS.

$V = 2100$ f. s.; $c = 0.9$; $\log C = 0.63752$.

X (yards.)	ϕ	$\Delta'\phi$	$\Delta''\phi$	T (seconds.)	Drift (yards.)	ω	Striking velocity.	τ (inches.)	Maximum ordinate. (feet.)
500	0° 19'.4	0'.1	1'.0	0.7	0.1	0° 21'	1999	17.3	
1000	0 39 .5	0 .3	2 .0	1.5	0.3	0 42	1904	16.4	
1500	1 02 .4	0 .7	3 .0	2.3	0.7	1 09	1812	15.5	21.6
2000	1 25 .8	1 .3	4 .1	3.2	1.3	1 38	1725	14.7	
2500	1 52 .0	2 .1	5 .3	4.0	2.1	2 11	1643	14.0	
3000	2 19 .4	3 .0	6 .7	5.0	3.2	2 49	1564	13.3	99.6
3500	2 48 .1	4 .3	8 .2	6.0	4.4	3 31	1490	12.6	
4000	3 19 .4	6 .0	9 .6	7.0	6.3	4 19	1419	11.9	
4500	3 52 .7	7 .9	11 .1	8.1	8.5	5 11	1356	11.3	265.5
5000	4 28 .9	10 .1	12 .7	9.2	11.1	6 09	1294	10.8	350.7
5500	5 07 .0	12 .9	14 .4	10.4	14.3	7 16	1238	10.3	442.8
6000	5 47 .9	16 .3	16 .4	11.7	18.1	8 29	1187	9.8	555.6
6500	6 31 .8	19 .9	18 .4	13.0	22.4	9 48	1142	9.4	696.7
7000	7 19 .6	23 .9	20 .4	14.4	27.9	11 13	1103	9.0	855.4
7500	8 09 .3	28 .4	22 .6	15.8	34.2	12 46	1072	8.7	1044
8000	9 01 .7	33 .4	25 .1	17.2	41.2	14 23	1047	8.5	1254
8500	9 59 .0	38 .5	27 .0	18.7	49.3	16 05	1027	8.3	1497
9000	10 57 .5	43 .7	30 .1	20.3	58.5	17 49	1011	8.2	1783

In computing the drift column the following data was employed: $n = 30$; $\mu = 0.64$; $\frac{\lambda}{h} = 0.31$; and $g = 32.16$. With these data and the given value of V, the drift formula reduces to

$$D = 0.68759 \left\{ \frac{(Bu) - 3.436}{z} - 0.00253 \right\} \frac{X}{\cos^3 \phi}.$$

The formula for the variation of ϕ due to a variation of one-tenth of the value of C reduces, for this gun, to

$$\varDelta'\phi = \mp \frac{746.1}{\cos 2\phi}(B - A),$$

and for the variation of ϕ due to a variation of $\pm$ 50 f. s. in the muzzle velocity the formula becomes

$$\varDelta''(\phi) = \mp \frac{7461}{\cos 2\phi}\varDelta_v.$$

If C and V both vary at the same time, the total variation of ϕ will be the sum of the partial variations, regard being had to the signs of these latter.

Finally, the expression for the thickness of armor the shots will penetrate at the various ranges becomes, by Maitland's formula, in inches,

$$\tau = 0.0091898v - 1.12,$$

or. if we prefer the French formula, we have, after reduction,

$$\tau = (0.003631v)^{\frac{100}{85}}.$$

The value of C was corrected for altitude for all ranges greater than 4000 yards, by the method fully explained in Prob. XIII.

Variation of the Angle of Departure due to a Variation of the Range.—This variation is found at a glance from the range table. In the absence of a range table we may compute the variation by the following simple formula, which is easily deduced from the principle of the rigidity of the trajectory:

$$\varDelta\phi = -3438 \tan\omega \frac{\varDelta X}{X}.$$

If the angle of fall is not known approximately, it may be computed by Prob. IX. It can generally be *estimated* with

sufficient accuracy for use in the above formula, by comparison with the angle of departure.

Example 2. For the 8-in. converted rifle we have, for a range of 2700 yards, $\phi = 5° \ 17'$ when the air is normal. With a head-wind of 30 miles per hour the range would be shortened by 50 yards, according to the method of Prob. VIII. How much should the angle of departure be increased in order that the range may be 2700 yards?

Here we have $X = 2700$, $\Delta X = -50$, and (suppose) $\omega = 6°$.

$$\therefore \Delta\phi = \frac{3438 \times 50}{2700} \tan 6° = 6\tfrac{3}{8} \text{ minutes of arc.}$$

If we had assumed $\omega = 6° \ 30'$ we should have found $\Delta\phi = 7'$.

Variation of the Range due to a Variation of the Muzzle Velocity.—For this variation we have, for a variation of $\pm$ 50 f. s. in the muzzle velocity,

$$\frac{\Delta X}{X} = \pm \frac{C\Delta_v}{2 \cos 2\phi \tan \omega}$$

in which Δ_v is taken from Table A.

Since, in direct fire, we have, very nearly,

$$2 \cos 2\phi \tan \omega = \sin 2\omega = BC,$$

the above expression for ΔX may be written

$$\frac{\Delta X}{X} = \pm \frac{\Delta_v}{B},$$

in which, it must be remembered, Δ_v is to be taken from Table A.

Example 3. In Ex. 1, Prob. IX, how much would the range be increased by increasing the muzzle velocity to 1450 f. s.?

Here $X = 4676$ feet; $\Delta_v = .0035$; and $B = .0681$.

$$\therefore \Delta X = \frac{.0035 \times 4676}{.0681} = 240 \text{ feet.}$$

APPENDIX I.

DEDUCTION OF THE GENERAL FORMULÆ OF DIRECT FIRE.

Resistance of the Air to the Motion of a Projectile.— A projectile leaves the gun in the direction of the axis of the bore, and with a given muzzle velocity; and is thenceforward, during its flight, constantly subjected to the action of two forces, which alone determine, in connection with the initial conditions, the curve, or trajectory, which its centre of gravity must describe. These forces are the constant force of gravity, which acts vertically downward, and the variable resistance of the air, which acts in a direction opposite to that of the motion of the projectile at each instant. This last is a tangential, retarding force, whose dynamical equivalent is, designating the resistance by ρ,

$$\rho = -M\frac{dv}{dt};$$

or, substituting the weight of the projectile for its mass,

$$\rho = -\frac{g}{w}\frac{dv}{dt}.$$

It has been proven by many conclusive experiments that the resistance encountered by a projectile at any point of its trajectory is directly proportional to the area exposed to resistance, and also to some function of the velocity with which it is moving at the time under consideration. If we assume that the axis of the projectile coincides with the tangent to the trajectory throughout its flight, which is very nearly correct in direct fire, the area exposed to resistance will be the area of the surface of the ogival head. This area, as is easily shown

by the integral calculus, varies as the square of the diameter of the projectile, that is, with d^2. We may, therefore, write the expression for the resistance

$$\rho = d^2 f(v).$$

With regard to $f(v)$ it is impossible with our present knowledge to determine its form. It has been shown, however, by Bashforth and other experimenters, that for velocities greater than 1330 f. s. and less than 790 f. s. the resistance varies very nearly as the square of the velocity; but that, for velocities between these limits, the resistances vary more rapidly than the square of the velocity.

Assuming

$$f(v) = \frac{A}{g} v^n,$$

in which A and n are to be determined by experiment, we have, for the expression for ρ,

$$\rho = \frac{Ad^2}{g} v^n,$$

and for the retardation,

$$\frac{g}{w}\rho = -\frac{dv}{dt} = \frac{Ad^2}{w} v^n = \frac{A}{C} v^n.$$

Oblong Projectiles.—A discussion of Bashforth's experiments made by the author in 1883 gave the following as the most probable values of A and n for oblong projectiles having ogival heads struck with radii of 1½ calibres:

Velocities greater than 1330 f. s.:

$$n = 2; \ \log A = 6.1525284 - 10.$$

1330 f. s. $> v >$ 1120 f. s.:

$$n = 3; \ \log A = 3.0364351 - 10.$$

1120 f. s. $> v >$ 990 f. s.:

$$n = 6; \ \log A = 3.8865079 - 20.$$

$990 \text{ f. s.} > v > 790 \text{ f. s.}$:

$$n = 3; \ \log A = 2.8754872 - 10.$$

$790 \text{ f. s.} > v > 100 \text{ f. s.}$:

$$n = 2; \ \log A = 5.7703827 - 10.$$

In applying these expressions for computing the resistance and retardation a projectile suffers, we must take d in inches, w in pounds and v in feet per second. The value of g is 32.16 f. s.

Example 1. Compute the resistance and retardation for the service projectile used with the 8-in. converted M. L. rifle, when it is moving with a velocity of 1350 f. s.

Here $d = 8$ in., $w = 183$ lbs., $c = 1$, $n = 2$ and $V = 1350$ f. s.

For the resistance we have

$$\begin{aligned} \log A &= 6.15253 \\ 2 \log d &= 1.80618 \\ 2 \log v &= 6.26066 \\ \text{a. c.} \log g &= 8.49268 \\ \hline \log \rho &= 2.71205 \qquad \therefore \rho = 515.3 \text{ lbs.} \end{aligned}$$

For the retardation $= \dfrac{g}{w}\rho$ we have

$$\begin{aligned} \log \rho &= 2.71205 \\ \log g &= 1.56732 \\ \text{a. c.} \log w &= 7.73755 \\ \hline \log \text{Ret.} &= 1.95692 \qquad \therefore \text{Ret.} = 90.56 \text{ f. s.} \end{aligned}$$

Example 2. Compute the resistance suffered by an 8-in. service projectile fired from the new navy rifle, when its velocity is 2000 f. s.

Here $d = 8$ in., $w = 250$ lbs., $c = 0.9$, $n = 2$ and $v = 2000$ f. s.

In this example we must write the expression for ρ as follows:

$$\rho = \frac{Acd^2}{g} v^n.$$

Performing the calculations as in Ex. 1, we find

$$\rho = 1131 \text{ lbs.};$$
$$\text{Retardation} = 145.5 \text{ f. s.}$$

That is, the resistance of 1131 lbs. would, if it remained constant for one second, diminish the velocity of the projectile by 145.5 f.s.

Terminal Velocity.—Let v_t be the velocity of a projectile when the resistance is equal to its weight. We must have in this case

$$\rho = \frac{Ad^2}{g} v_t^n = w,$$

whence

$$v_t = \left(\frac{gw}{Ad^2}\right)^{\frac{1}{n}}.$$

The velocity v_t is called *terminal velocity*, because it is easily seen to be the velocity toward which a body, falling in a resisting medium like the air, continually approaches, and only reaches at infinity.

Example 3. What is the terminal velocity of the projectile of Ex. 2, supposing it to fall point downward?

It will be found by trial that the terminal velocity lies between 1120 and 990 f. s.; and, therefore, $n = 6$. Therefore

$$\begin{aligned} \log g &= 1.50732 \\ \log w &= 2.39794 \\ \text{a. c. } \log d^2 &= 8.19382 \\ \text{a. c. } \log A &= \underline{16.11349} \\ & 6\overline{)18.21257} \\ \log v_t &= 3.03543 \qquad \therefore v_t = 1085 \text{ f. s.} \end{aligned}$$

Similarly, it may be shown that the terminal velocity of the projectile of Ex. 1 is 1030 f. s.

Spherical Projectiles.—General Mayevski, from a discussion of his own experiments with spherical projectiles made at

St. Petersburg in 1868, deduced values for A and n which, reduced to English units, are as follows:

Velocities greater than 1233 f. s.:

$$\rho = \frac{Ad^2}{g} v^2 ; \log A = 6.3088473 - 10.$$

Velocities less than 1233 f. s.:

$$\rho = \frac{Ad^2}{g} v^2 \left(1 + \frac{v^2}{r^2}\right) ; \log A = 5.6029333 - 10 ; r = 610.25.$$

Example 4. Required the diameter and weight of a solid, spherical, cast-iron shot whose terminal velocity in air shall be 1233 f. s.

We have

$$\rho = w = \frac{Ad^2 v^2}{g},$$

whence

$$\frac{d^2}{w} = \frac{g}{Av^2}.$$

Let d_1 be the diameter of a similar shot whose weight (w_1) is known. Then

$$\frac{d^2}{w} = \frac{d_1}{w_1},$$

whence, by division,

$$d = \frac{Ad_1^3 v^2}{gw_1}.$$

The solid cast-iron shot for the 15-in. S. B. gun is 14.87 in. in diameter and weighs 450 lbs. Therefore, substituting in the above equation, we find $d = 70$ inches. We also have

$$w = \frac{d^3 w_1}{d_1^3} = 59900 \text{ lbs.}$$

We see from this example that the terminal velocities of all service spherical projectiles are less than 1233 f. s.

Solving the equation

$$\frac{Ad^2}{g} v^2\left(1 + \frac{v^2}{r^2}\right) = w$$

for v, we obtain

$$v_t = \sqrt{\frac{r^2}{2}\left(\sqrt{1 + \frac{4wg}{Ad^2r^2}} - 1\right)},$$

by means of which the terminal velocities of service solid shot may be computed.

Differential Equations of Motion.—We will assume that the projectile, if spherical, has no motion of rotation; and, in addition to this, in the case of oblong projectiles, that the axis of the projectile lies constantly in the tangent to the trajectory; also, that the air through which it moves is still and of uniform density. As none of these conditions is ever exactly fulfilled in practice, the equations deduced will only give what may be called the *normal trajectory*, or the trajectory in the plane of fire, and from which the actual trajectory will deviate more or less. It is evident, however, that this deviation from the plane of fire is relatively small; that is, small in comparison with the whole extent of the trajectory, owing to the very great density of the projectile as compared with that of the air.

Let the muzzle of the gun be taken as the origin of rectangular co-ordinates, of which let the axis of X be horizontal and that of Y vertical. The retardation at any point of the trajectory whose co-ordinates are x and y, and at which the inclination of the tangent to the horizon is θ, in the direction of the tangent, due to the resistance of the air, is, as we have seen, $\frac{w}{g}\rho$; and the corresponding retardation due to the action of gravity is $g \sin \theta$. Therefore, the total retardation in the direction of motion is expressed by the equation

$$\frac{dv}{dt} = -\frac{g}{w}\rho - g \sin \theta.$$

The first term of the second member of this last equation is always negative, since the resistance of the air tends to reduce the velocity. The second term is negative in the ascending branch; but in the descending branch $\sin\theta$ changes its sign and the term becomes positive. The reasons are evident.

The velocities parallel to X and Y are, respectively, $v\cos\theta$ and $v\sin\theta$; and the accelerations parallel to the same axes are $\frac{g}{w}\rho\cos\theta$ and $g+\frac{g}{w}\rho\sin\theta$. Therefore,

$$\frac{d(v\cos\theta)}{dt}=-\frac{g}{w}\rho\cos\theta, \quad \ldots \ldots \quad (1')$$

and

$$\frac{d(v\sin\theta)}{dt}=-g-\frac{g}{w}\rho\sin\theta. \quad \ldots \ldots \quad (2')$$

Performing the differentiations indicated in these equations, then multiplying the first by $\sin\theta$ and the second by $\cos\theta$, and taking the difference of the products, gives

$$\frac{vd\theta}{dt}=-g\cos\theta. \quad \ldots \ldots \quad (3')$$

Since the resistance of the air does not enter into this last equation, it must be an expression for the forces resolved normally to the trajectory, that is, at right angles to the direction of the resistance; as may otherwise be easily shown.

Designate the horizontal velocity by v_1; that is, let

$$v_1=v\cos\theta.$$

Introducing this into (1′) and (3′), they become

$$\frac{dv_1}{dt}=-\frac{g}{w}\rho\cos\theta, \quad \ldots \ldots \quad (4')$$

and

$$\frac{v_1d\theta}{dt}=-g\cos^2\theta; \quad \ldots \ldots \quad (5')$$

whence, eliminating dt,

$$\frac{dv_1}{d\theta} = \frac{\rho}{w} \frac{v_1}{\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6')$$

Eq. (4′) gives

$$dt = -\frac{dv_1}{\frac{g}{w}\rho\cos\theta};$$

but, as we have already seen, page 258,

$$\frac{g}{w}\rho = \frac{Av^n}{C} = \frac{Av_1^n}{C\cos^n\theta};$$

whence, by substitution,

$$dt = -\frac{C}{A}\cos^{n-1}\theta\frac{dv_1}{v_1^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7')$$

The relations between the element of time and the elements of the trajectory at any point are given by the fundamental equations of mechanics, $dx = v_1 dt$; $dy = v_1 \tan\theta dt$; and $ds = v_1 \sec\theta d\theta$ (s being the length of the trajectory from the origin). Substituting for dt in these equations its value from (7′), they become

$$dx = -\frac{C}{A}\cos^{n-1}\theta\frac{dv_1}{v_1^{n-1}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8')$$

$$dy = -\frac{C}{A}\sin\theta\cos^{n-2}\theta\frac{dv_1}{v_1^{n-1}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

$$ds = -\frac{C}{A}\cos^{n-2}\theta\frac{dv_1}{v_1^{n-1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10')$$

We may also make θ the independent variable as follows: From (5′) we have

$$dt = -\frac{v_1}{g}\frac{d\theta}{\cos^2\theta} = -\frac{v_1}{g}d\tan\theta; \quad . \quad . \quad . \quad . \quad . \quad (11')$$

whence, substituting as before,

$$dx = -\frac{v_1^2}{g} d \tan \theta ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12')$$

$$dy = -\frac{v_1^2}{g} \tan \theta d \tan \theta ; \quad . \quad . \quad . \quad . \quad . \quad (13')$$

$$ds = -\frac{v_1^2}{g} \sec \theta d \tan \theta . \quad . \quad . \quad . \quad . \quad . \quad (14')$$

The horizontal velocity may be eliminated from these last four equations in the following manner: We have, by hypothesis,

$$\frac{\rho}{w} = \frac{Av^n}{gC} = \frac{Av_1^n}{gC \cos^n \theta},$$

which, substituted in (6′), gives

$$\frac{d\theta}{\cos^{n+1} \theta} = \frac{gC}{A} \frac{dv_1}{v_1^{n+1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15')$$

Both members of (15′) can be integrated in finite terms when n is any whole number. Symbolizing the integral of the first member by (θ), that is, making

$$(\theta) = \int \frac{d\theta}{\cos^{n+1} \theta},$$

and integrating between the limits ϕ and θ, to which correspond in the second member V_1 and v_1, we have

$$(\phi) - (\theta) = \frac{gC}{nA} \left\{ \frac{1}{v_1^n} - \frac{1}{V_1^n} \right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

Let i be the value of ϕ when V_1 is infinite; or, what is the same thing, let

$$(i) = \frac{gC}{nAv_1^n} + (\theta);$$

then we have

$$(i) - (\theta) = \frac{gC}{nAv_1^n}.$$

Making, for simplicity,

$$k^n = \frac{gC}{nA},$$

and solving for v_1, we have

$$v_1 = \frac{k}{\{(i) - (\theta)\}^{\frac{1}{n}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17')$$

Substituting this value of v_1 in (11′), (12′), (13′), and (14′), they become, respectively,

$$dt = -\frac{k}{g}\frac{d \tan \theta}{\{(i) - (\theta)\}^{\frac{1}{n}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (18')$$

$$dx = -\frac{k^2}{g}\frac{d \tan \theta}{\{(i) - (\theta)\}^{\frac{2}{n}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (19')$$

$$dy = -\frac{k^2}{g}\frac{\tan \theta d \tan \theta}{\{(i) - (\theta)\}^{\frac{2}{n}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (20')$$

$$ds = -\frac{k^2}{g}\frac{\sec \theta d \tan \theta}{\{(i) - (\theta)\}^{\frac{2}{n}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21')$$

Either of the three groups of equations which have been deduced above may be said to contain the whole theory of the motion of a projectile in the plane of fire when subjected to the resistance of a medium whose action can be expressed as a function of the velocity. But, unfortunately, the laws of resistance which admit of the complete integration of these differential equations are very few, and do not include any which are found to belong to our atmosphere—at least, not for the high velocities employed in gunnery.

Cases which Admit of Integration in Finite Terms.—The following are all the cases which can be solved in finite terms; that is, in terms of known functions:

1. As has already been shown, the velocity can be determined in terms of θ when n is *any integer.*

2. When the resistance is supposed to be *zero*, that is, *in vacuo*, all the equations of motion can be integrated, and θ can be eliminated between the expressions for x and y, as will be shown in the next section.

3. When the resistance is considered *constant* for all velocities we can deduce expressions for v, t, x, y, and s in terms of θ.

4. When $n = 1$, all the elements except s may be expressed in terms of θ, and also of x. The equation to the trajectory in this case is

$$y = \left(\frac{k}{V_1} + \tan \phi\right)x + \frac{k^2}{g}\log_\epsilon\left(1 - \frac{gx}{kV_1}\right).$$

5. The expression for ds can be integrated when the resistance varies as the square of the velocity, as will be shown further on.

6. Professor Greenhill has recently succeeded in deducing expressions for the elements of a trajectory in terms of elliptic functions when $n = 3$.

Motion in Vacuo.—Making $\rho = 0$, Eq. (4) becomes

$$dv_1 = 0;$$

and, therefore, v_1, or the horizontal velocity, is in this case constant and equal to its initial value V_1. Therefore, *in vacuo*,

$$v \cos \theta = V \cos \phi.$$

Integrating (11′), (12′) and (13′) between the limits ϕ and θ gives, if $v_1 = V_1$,

$$t = \frac{V_1}{g}(\tan \phi - \tan \theta); \quad . \quad . \quad . \quad . \quad (22')$$

$$x = \frac{V_1^2}{g}(\tan\phi - \tan\theta); \quad . \quad . \quad . \quad . \quad (23')$$

$$y = \frac{V_1^2}{2g}(\tan^2\phi - \tan^2\theta). \quad . \quad . \quad . \quad . \quad (24')$$

Eq. (14′) becomes, when $v_1 = V_1$,

$$ds = -\frac{V_1^2}{g}\frac{d\theta}{\cos^3\theta}.$$

Making

$$(\theta) = \int \frac{d\theta}{\cos^3\theta} = \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\log_e(\tan\theta + \sec\theta),$$

we have

$$s = \frac{V_1^2}{g}((\phi) - (\theta)). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25')$$

Eliminating $\tan\theta$ from (23′) and (24′) by division and addition, we have

$$y = x\tan\phi - \frac{gx^2}{2V_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26')$$

which is the equation of a parabola whose axis is vertical. A parabola is, therefore, the curve a projectile would describe *in vacuo.*

Since a parabola is symmetrical with respect to its axis, the descending branch of the trajectory *in vacuo* is similar in every respect to the ascending branch, and the angle of fall is equal to the angle of projection, but with the opposite sign; and, generally, for the same value of y in the two branches, $\tan\theta$ has the same value numerically, but with contrary signs; being positive in the ascending branch, negative in the descending branch, and zero at the summit.

For the whole range we evidently have $x = X$, and $\theta = -\phi$. Making these substitutions in (23′), we have, for the range,

$$X = \frac{2V_1^2}{g}\tan\phi = \frac{V^2\sin 2\phi}{g}.$$

Subtracting (23′) from this last equation and reducing gives

$$X - x = \frac{X}{2 \tan \phi}(\tan \phi + \tan \theta).$$

Also, dividing (24′) by (23′) gives

$$\frac{y}{x} = \tfrac{1}{2}(\tan \phi + \tan \theta);$$

whence

$$y = \frac{x}{X}(X - x) \tan \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27')$$

Making $\theta = -\phi$ in (22) gives the following expression for the time of flight:

$$T = \frac{2V_1}{g} \tan \phi = \frac{2V}{g} \sin \phi.$$

Subtracting (22′) from this last equation gives

$$T - t = \frac{V_1}{g}(\tan \phi + \tan \theta);$$

also, (24′) divided by (22′) gives

$$\frac{y}{t} = \frac{V_1}{2}(\tan \phi + \tan \theta);$$

whence

$$y = \frac{gt}{2}(T - t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28')$$

For an application of (28′) see page 131.

All the properties of the trajectory *in vacuo* may be easily and elegantly determined by means of the fundamental Eqs. (16′) to (19′) inclusive.

Trajectory in the Air. Approximate Equations of Motion for Direct Fire.—Eqs. (15′), (7′) and (8′) may be written, respectively,

$$\frac{d\theta}{\cos^2\theta} = \frac{gC}{A\sec^{n-1}\theta}\frac{dv_1}{v_1^{n+1}};$$

$$dt = -\frac{C}{A\sec^{n-1}\theta}\frac{dv_1}{v_1^{n}};$$

$$dx = -\frac{C}{A\sec^{n-1}\theta}\frac{dv_1}{v_1^{n-1}}.$$

In these three equations the first members are exact integrals; and the same would be true of the second members except for the variable factor $\frac{1}{\sec^{n-1}\theta}$ which enters into each of them. In all examples of direct fire, $\sec\theta$ differs but little from unity; and its mean value for the entire trajectory above the level of the gun evidently lies between unity and $\sec\omega$. We might, for the smaller and more important angles of departure of direct fire, take unity for its mean value without introducing any material error into the resulting equations. (See Probs. I to VIII.) A still nearer approximation results from making

$$\sec^{n-1}\theta = \sec^{n-2}\phi;$$

by which substitution the above équations become, by slight reductions,

$$\frac{d\theta}{\cos^2\theta} = \frac{gC}{A\cos^2\phi}\frac{d(v_1\sec\phi)}{(v_1\sec\phi)^{n+1}};$$

$$dt = -\frac{C}{A\cos\phi}\frac{d(v_1\sec\phi)}{(v_1\sec\phi)^{n}};$$

$$dx = -\frac{C}{A}\frac{d(v_1\sec\phi)}{(v_1\sec\phi)^{n-1}}.$$

Integrating between the limits ϕ and θ, and making

$$v_1 \sec\phi = \frac{v\cos\theta}{\cos\phi} = u,$$

and

$$V_1 \sec\phi = \frac{V\cos\phi}{\cos\phi} = V,$$

we have

$$\tan\phi - \tan\theta = \frac{gC}{nA\cos^2\phi}\left\{\frac{1}{u^n} - \frac{1}{V^n}\right\};$$

$$t = \frac{C}{(n-1)A\cos\phi}\left\{\frac{1}{u^{n-1}} - \frac{1}{V^{n-1}}\right\};$$

$$x = \frac{C}{(n-2)A}\left\{\frac{1}{u^{n-2}} - \frac{1}{V^{n-2}}\right\}.$$

When $n = 2$ the above expression for x becomes indeterminate. But when $n = 2$, we have

$$dx = -\frac{C}{A}\frac{du}{u},$$

and, therefore,

$$x = \frac{C}{A}\{\log_e V - \log_e u\}.$$

The expressions for $\tan\theta$, t, and x may be still further simplified by the following substitutions. In the first, make

$$I(u) = \frac{2g}{nAu^n} + Q,$$

and, similarly,

$$I(V) = \frac{2g}{nAV^n} + Q.$$

With these substitutions we have

$$\tan\phi - \tan\theta = \frac{C}{2\cos^2\phi}\{I(u) - I(V)\};$$

which is Eq. (4) of the text. The arbitrary constant Q, introduced into the expressions for $I(u)$ and $I(V)$, is necessary in computing the tables of these functions. Its presence causes no difficulty, since it disappears from the formula by subtraction. The functions $I(u)$ and $I(V)$ are called *inclination-functions.*

In the expression for t, make

$$T(u) = \frac{1}{(n-1)Au^{n-1}} + Q;$$

and similarly for V. We then have for the time, expressed in *time-functions,*

$$t = \frac{C}{\cos\phi}\{T(u) - T(V)\};$$

the same as Eq. (2) of the text.

For the abscissa x, make

$$S(u) = \frac{1}{(n-2)Au^{n-2}} + Q,$$

or

$$S(u) = -\frac{\log_e u}{A} + Q,$$

according as n is greater than, or equal to, 2. We then have for x, expressed in *space-functions,*

$$x = C\{S(u) - S(V)\},$$

as in Eq. (1) of the text.

To deduce an expression for the altitude of the projectile at any point of its flight, above the level of the gun, we proceed as follows: We have from (4) (already deduced), since $\tan\theta = \frac{dy}{dx}$,

$$\frac{dy}{dx} = \tan\phi - \frac{C}{2\cos^2\phi}\{I(u) - I(V)\},$$

or

$$\frac{2\cos^2\phi}{C}\left\{\frac{dy}{dx}-\tan\phi\right\}-I(V)=-I(u).$$

We also have from the differential expression for x,

$$\frac{dx}{C}=-\frac{du}{Au^{n-1}};$$

whence, multiplying the last two equations together, member by member,

$$\frac{2\cos^2\phi}{C^2}\{dy-\tan\phi dx\}-\frac{I(V)}{C}dx=\frac{I(u)du}{Au^{n-1}}.$$

Integrating, and making x and y both zero at the origin, where $u = V$, we have

$$\frac{2\cos^2\phi}{C^2}\{y-x\tan\phi\}-\frac{I(V)}{C}x=-\frac{1}{A}\int_u^V\frac{I(u)du}{u^{n-1}}.$$

Symbolizing the second member of this equation by $A(u)$ (*altitude-function*), and observing the limits, we have

$$\frac{2\cos^2\phi}{C^2}\{y-x\tan\phi\}-\frac{I(V)}{C}x=-\{A(u)-A(V)\}.$$

From the expression for x we have

$$\frac{X}{C}=S(u)-S(V);$$

whence, by division,

$$\frac{2\cos^2\phi}{C}\left\{\frac{y}{x}-\tan\phi\right\}-I(V)=-\frac{A(u)-A(V)}{S(u)-S(V)};$$

or, finally,

$$\frac{y}{x}=\tan\phi-\frac{C}{2\cos^2\phi}\left\{\frac{A(u)-A(V)}{S(u)-S(V)}-I(V)\right\},$$

which is Eq. (3) of the text.

To deduce a working expression for the altitude-functions, we have, since

$$I(u) = \frac{2g}{nAu^n} + Q,$$

$$A(u) = -\frac{2g}{nA^2}\int\frac{du}{u^{2n-1}} - \frac{Q}{A}\int\frac{du}{u^{n-1}} + Q'$$
$$= \frac{g}{n(n-1)A^2u^{2(n-1)}} + \frac{Q}{(n-2)Au^{n-2}} + Q';$$

which becomes, when $n = Q2$,

$$A(u) = \frac{g}{2A^2u^2} - \frac{Q}{A}\log_e u + Q'.$$

By differentiating the expressions we have given for the space, altitude, inclination, and time functions, we obtain the following equations, which are sometimes useful (see page 249):

$$\frac{dS(u)}{du} = -\frac{1}{Au^{n-1}};$$

$$\frac{dA(u)}{du} = -\frac{I(u)}{Au^{n-1}};$$

$$\frac{dI(u)}{du} = -\frac{2g}{Au^{n+1}};$$

$$\frac{dT(u)}{du} = -\frac{1}{Au^n} = -\frac{d^2}{g\rho}.$$

To reduce Mayevski's *drift-functions* $M(u)$ and $B(u)$, as given on page 181, to forms suitable for computation, we proceed as follows: We have

$$M(u) = -\int\frac{du}{u^2f(u)} = -\frac{1}{A}\int\frac{du}{u^{n+2}},$$

since

$$f(u) = Au^n.$$

Therefore

$$M(u) = \frac{1}{(n+1)Au^{n+1}} + Q.$$

Also,

$$B(u) = -\int M(u)\frac{udu}{f(u)};$$

whence, by substituting for $M(u)$ and $f(u)$ their values, we have

$$B(u) = -\frac{1}{(n+1)A^2}\int\frac{du}{u^{2n}} - \frac{Q}{A}\int\frac{du}{u^{n-1}} + Q'$$

$$= \frac{1}{(n+1)(2n-1)A^2u^{2n-1}} + \frac{Q}{(n-2)Au^{n-2}} + Q',$$

which becomes when $n = 2$,

$$B(u) = \frac{1}{9A^2u^3} - \frac{Q}{A}\log_e u + Q'.$$

The values of the six ballistic functions which have been investigated in the preceding pages will be found tabulated in Table I. In their computation the values of n and A deduced from Bashforth's experiments were employed. They are given on page 258.

Expression for the Velocity when the Resistance varies as the Square of the Velocity.—As has already been stated, the velocity of a projectile can always be determined in exact terms, when the law of resistance is that of an integral power of the velocity. For low velocities such as are employed in curved and high-angle fire, the law of resistance is very approximately that of the square of the velocity; and the same law holds for high velocities down to about 1330 f. s. For these velocities a simple and useful formula may be deduced from (16′), giving the relation between the horizontal velocities at any two points of a trajectory and the corresponding inclinations.

Making $n = 2$ in (16′), it becomes

$$(\phi) - (\theta) = \frac{gC}{2A}\left\{\frac{1}{v_1^2} - \frac{1}{V_1^2}\right\},$$

in which

$$(\theta) = \int \frac{d\theta}{\cos^3\theta} = \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\log_e(\tan\theta + \sec\theta).$$

But $\dfrac{g}{Av_1^2}$ is the inclination-function when $n = 2$ (see page 271); and, therefore,

$$(\phi) - (\theta) = \frac{C}{2}\{I(v_1) - I(V_1)\};$$

or

$$I(v_1) = \frac{2}{C}\{(\phi) - (\theta)\} + I(V_1); \quad . \quad . \quad (29')$$

by means of which v_1 can be easily and accurately determined at any point of the trajectory for which θ is known. The values of the function (θ) will be found tabulated in Table IV.

From the expression for (θ) it will readily be seen that the function is 0 when $\theta = 0$, negative when θ is negative, and infinite when $\theta = \frac{1}{2}\pi$; or, in symbols, $(0) = 0$, $(-\theta) = -(\theta)$, and $(\frac{1}{2}\pi) = \infty$. At the summit of the trajectory, therefore, where $\theta = 0$, we have

$$I(v_0) = \frac{2(\phi)}{C} + I(V_1),$$

an equation analogous to (6).

By means of Eq. (29′) we may illustrate the accuracy of the general formulæ we have deduced for direct fire in assuming that

$$\sin^{n-1}\theta = \sin^{n-2}\phi,$$

and by the resulting change of v_1 into $v_1 \sec\phi$. For this purpose take the following examples:

Example 1. Compute the summit velocity of the projectile

of Ex. 1, Prob. XXII, when $\phi = 10°$. (*a*) By Eq. (29′); (*b*) by Eqs. (6) and (7).

(*a*) Eq. (29′) becomes in this case, since $\theta = 0°$,

$$I(v_0) = \frac{2(10°)}{C} + I(2100 \cos 10°)$$

$$= \frac{2 \times 0.17724}{C} + I(2068.1) = 0.10573;$$

$$\therefore v_0 = 1296.6 \text{ f. s.}$$

(*b*) We will now compute the value of v_0 by the approximate equations

$$I(u_0) = \frac{\sin 2\phi}{C} + I(V)$$

and

$$v_0 = u_0 \cos \phi.$$

We have

$$I(u_0) = \frac{\sin 20°}{C} + I(2100) = 0.10126,$$

which gives $u_0 = 1320.0$;

$$\therefore v_0 = 1320 \cos 10° = 1300.0 \text{ f. s.},$$

an error of only 3.4 f. s.

Example 2. Given $V = 790$ f. s.; $\phi = 10°$ and $C = 1$. Compute v_ω by both the preceding methods.

We must first compute the final velocity and angle of fall by the second method of Prob. X.

We have, first,

$$I(u_0) = \sin 20° + I(790) = 0.85988.$$

Next,

$$\frac{A(u_\omega) - 1856.71}{S(u_\omega) - 11258.8} = 0.85988;$$

from which we find, by trial, $u_\omega = 577.23$. Then,

$$\tan \omega = \frac{I(u_\omega) - I(u_0)}{2 \cos^2 \phi};$$

$$\therefore \omega = 12^\circ\ 15'\ 20''.$$

Lastly,

$$v_\omega = u \frac{\cos \phi}{\cos \omega} = 581.7 \text{ f. s.}$$

By the *analytically* exact method we have

$$I(v_1) = 2\{(10^\circ) + (12^\circ\ 15'\ 20'')\} + I(790 \cos 10^\circ) = 1.32736;$$

$$\therefore v_1 = 569.3 \text{ f. s.},$$

and

$$v_\omega = v_1 \sec 12^\circ\ 15'\ 20'' = 582.5 \text{ f. s.}$$

We may also compare the velocities at the summit in this example, by both methods, with very little additional labor. We have found in the first case $I(u_0) = 0.85988$, which gives $u_0 = 669.78$. Therefore

$$v_0 = 669.78 \cos 10^\circ = 659.6 \text{ f. s.}$$

In the second case we have

$$I(v_0) = 2(10^\circ) + I(790 \cos 10^\circ) = 0.89952;$$

$$\therefore v_0 = 659.1.$$

Some writers on ballistics ignore the relation

$$v_0 = u \frac{\cos \phi}{\cos \theta},$$

and assume that

$$v = u$$

in all cases. A careful consideration of the above discussion would seem to show that this cannot be done safely except in cases where the ratio

$$\frac{\cos \phi}{\cos \theta}$$

is, practically, unity. It will also show that the approximate formulæ for direct fire are more exact than is generally supposed.

In order still further to test the analytical accuracy of the general formulæ for direct fire, we will consider the following example:

Example 3. Given $V = 1886$ f. s., $\phi = 10°$, $d = 12$ in., $w = 800$ lbs., and $C = \frac{800}{144} = \frac{100}{18}$, to compute v, x, y, and t for three points of the trajectory at which $\theta = 0°$, $\theta = -9°$, and $\theta = -13°$, respectively, (*a*) by a method analytically exact, and (*b*) by the general formulæ of direct fire.

(*a*) We will first compute the summit velocity by (29′), as follows:

$$I(v_0) = \frac{2(10°)}{C} + I(1857.35) = 0.10055\,;$$

$$\therefore v_0 = 1322.5 \text{ f. s.}$$

As this value of v_0 differs but slightly from 1330 f. s., we see that the law of resistance for the arc $(10° - 0°)$ is that of the square of the velocity. Making $n = 2$ in (18′) and integrating, it becomes

$$t = \frac{k}{g}\int_\theta^\phi \frac{\sec^2\theta d\theta}{\{(i) - (\theta)\}^{\frac{1}{2}}} = \frac{k}{g}{}_\phi T^\theta\,;$$

representing (as Bashforth has done) the definite integral by ${}_\phi T^\theta$. Similarly, for the other elements we have

$$x = \frac{k^2}{g}{}_\phi X^\theta\,;$$

$$y = \frac{k^2}{g}{}_\phi Y^\theta\,;$$

$$s = \frac{k^2}{g}{}_\phi S^\theta.$$

The values of the above definite integrals from $\phi = 10°$ to $\theta = 0°$, computed with great care by Weddle's quadrature formula, are as follows:

$$
\begin{aligned}
{}^{10°}T^{0°} &= 0.34317; \\
{}^{10°}X^{0°} &= 0.67507; \\
{}^{10°}Y^{0°} &= 0.06607; \\
{}^{10°}S^{0°} &= 0.67917.
\end{aligned}
$$

The value of k is computed by the formula

$$k^2 = \frac{gC}{2A},$$

employing the value of A given on page 258 for velocities greater than 1330 f. s.

The results of the calculations are as follows:

$$
\begin{aligned}
{}_{10°}t_{0°} &= 8''.4612; \\
{}_{10°}x_{0°} &= 13198 \text{ feet}; \\
{}_{10°}y_{0°} &= 1292 \quad \text{"} \\
{}_{10°}s_{0°} &= 13278 \quad \text{"}
\end{aligned}
$$

We may verify the correctness of s as follows: From (10′) we have, when $n = 2$,

$$ds = -\frac{C}{A}\frac{dv_1}{v_1};$$

whence

$$
\begin{aligned}
s &= \frac{C}{A}\{\log V_1 - \log v_1\} \\
&= C\{S(v_1) - S(V_1)\}. \quad . \quad . \quad . \quad . \quad . \quad (30')
\end{aligned}
$$

This equation gives the length of arc described by the projectile from the origin, when the resistance varies as the square of the velocity, and is analytically exact.

Applying numbers already known, we have

$${}_{10°}s_{0°} = \tfrac{100}{18}\{S(1322.5) - S(1857.35)\} = 13277.2 \text{ feet}.$$

This value of s differs from that computed by quadratures by less than one foot. We may therefore assume the correctness of the above computed values of t, x, and y, so far as the *formulæ* are concerned, and that is all we are interested in at present.

We will now compute t, x, and y for the arc $(10° - 0°)$ by the approximate formulæ of direct fire. We have for v,

$$I(u_0) = \tfrac{18}{100} \sin 20° + I(1886) = 0.09633\,;$$

$$\therefore u_0 = 1344.4\,;$$

$$\therefore v_0 = 1344.4 \cos 10° = 1324 \text{ f. s.}$$

For x_0 we have

$$x = \tfrac{100}{18}\{S(1344.4) - S(1886)\} = 13236 \text{ feet.}$$

To compute y_0 we employ the formula given in Prob. XV, page 113:

$$y = \frac{C^2}{2\cos^2\phi}\{I(u_0)z + A(V) - A(u_0)\} = 1297 \text{ feet.}$$

For t_0 we have

$$t_0 = \frac{100}{18 \cos 10°}\{T(1344.4) - T(1886)\} = 8''.4788.$$

From the above calculations we may infer that, for high velocities, the general formulæ of direct fire are practically correct for angles of departure up to 10°, the errors amounting to less than *one third of one per cent.* To illustrate the accuracy of the formulæ when the law of resistance is that of the cube of the velocity, we will continue this example from $\theta = 0°$, where the velocity is 1322.5 f. s., to $\theta = -9°$, and velocity $= 1127.56$ f. s.

Without going into details, the results are as follows:

	By Quadratures.	By the General Formulæ.
${}_0t_{-9^\circ}$	$5''.9424$	$5''.9999$
${}_0x_{-9^\circ}$	7187.6 ft.	7191.1 ft.
${}_0y_{-9^\circ}$	− 536.77 ft.	− 536.6 ft.
v_{-9°	1127.56 f. s.	1128.9 f. s.

For the arc extending from $\theta = -9^\circ$ and velocity $= 1127.56$ f. s., to $\theta = -13^\circ$ and velocity $= 1081.55$ f. s., the resistance varies as the sixth power of the velocity. The values of the elements for this arc computed by quadratures and by the approximate formulæ are as follows:

	By Quadratures.	By the General Formulæ.
${}_{-9^\circ}t_{-13^\circ}$	$2''.4390$	$2''.4412$
${}_{-9^\circ}x_{-13^\circ}$	2640.1 ft.	2645.6 ft.
${}_{-9^\circ}y_{-13^\circ}$	− 512.09 ft.	− 513.17 ft.
v_{-13°	1081.55 f. s.	1083.8 f. s.

For the whole arc, extending from $\phi = 10^\circ$ to $\theta = -13^\circ$, we have by quadratures, taking the sum of the several separate results, the following values, to which are added the same elements computed at *one operation* by the approximate formulæ:

	By Quadratures.	By the General Formulæ.
${}_{10^\circ}t_{-13^\circ}$	$16''.84$	$16''.87$
${}_{10^\circ}x_{-13^\circ}$	23026 ft.	23067 ft.
${}_{10^\circ}y_{-13^\circ}$	243 ft.	248 ft.
v_{-13°	1082 f. s.	1083 f. s.

The complete horizontal range, time of flight, etc., in this example, computed by the first method of Prob. IX, are as follows:

$$X = 24057 \text{ feet};$$

$$T = 17''.81;$$

$$\omega = 14^\circ\ 35';$$

$$v_\omega = 1072 \text{ f. s.};$$

which may be regarded as very close approximations from an analytical point of view.

Effect of the Wind upon the Range and Striking Velocity of a Projectile fired with a Small Angle of Elevation.— In the following discussion it will be assumed that the trajectory is so slightly inclined to the horizon that the horizontal velocity of the projectile is practically the same as its real velocity, and also that the motion of the wind is horizontal and uniform during the flight of the projectile. The method is therefore only applicable to the flatter trajectories of direct fire, say for angles of projection not exceeding about 8°.

We shall also assume that the effect of a wind blowing parallel to the range is simply to increase or diminish the resistance the projectile encounters. That is, if a projectile is moving nearly horizontally with a velocity v, the resistance of the air, if there is no wind, is proportional to v^n; but if the air has a horizontal motion W_p parallel to the plane of fire, then the resistance will be proportional to $(v + W_p)^n$ or $(v - W_p)^n$, according to the direction of W_p.

Upon this hypothesis the expression for the retardation (page 258) becomes

$$\frac{dv}{dt} = -\frac{A}{C}(v \pm W_p)^n;$$

whence, considering the motion horizontal, we have

$$dt = -\frac{C}{A}\frac{dv}{(v \pm W_p)^n};$$

and therefore, since $dx = vdt$,

$$dx = -\frac{C}{A}\frac{vdv}{(v \pm W_p)^n}.$$

The integration of the first equation between the limits V and v gives

$$T = C\left\{\frac{1}{(n-1)A(v \pm W_p)^{n-1}} - \frac{1}{(n-1)A(V \pm W_p)^{n-1}}\right\},$$

or

$$T = C\{T(v \pm W_p) - T(V \pm W_p)\},$$

as in Prob. VI.

The expression for dx given on the preceding page may be written

$$\begin{aligned} dx &= -\frac{C}{A}\frac{(v \pm W_p)dv \mp W_p dv}{(v \pm W_p)^n} \\ &= -\frac{C}{A}\left\{\frac{dv}{(v \pm W_p)^{n-1}} \mp \frac{W_p dv}{(v \pm W_p)^n}\right\} \\ &= -\frac{C}{A}\frac{dv}{(v \pm W_p)^{n-1}} \mp W_p dt. \end{aligned}$$

Integrating between the limits V and v, to which correspond $x = 0$ and $x = X$, and also $t = 0$ and $t = T$, we have

$$X = C\left\{\frac{1}{(n-2)A(v \pm W_p)^{n-2}} - \frac{1}{(n-2)A(V \pm W_p)^{n-2}}\right\} \mp W_p T;$$

or, finally,

$$X = C\{S(v \pm W_p) - S(V \pm W_p)\} \mp W_p T,$$

as in Prob. VII.

APPENDIX II.

FORMULÆ FOR MORTAR FIRING.

Euler's Method.—For practical muzzle velocities less than 800 f. s., the law of resistance is very nearly that of the square of the velocity. In this case $n = 2$, and (21′) becomes

$$ds = -\frac{C}{2A}\frac{\sec^3\theta d\theta}{(i)-(\theta)};$$

but, by definition, when $n = 2$,

$$(\theta) = \int \sec^3 \theta d\theta = \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\log_e(\tan\theta + \sec\theta);$$

and, therefore,

$$ds = -\frac{C}{2A}\frac{d(\theta)}{(i)-(\theta)}.$$

In this equation (i) is constant for a given trajectory, and is defined by either of the equations,

$$(i) = C\frac{k^2}{V_1^2} + (\phi) = C\frac{k^2}{v_1^2} + (\theta) = C\frac{k^2}{v_0^2},$$

in which

$$k^2 = \frac{g}{2A}.$$ (See page 265.)

Integrating the above expression for ds between the limits ϕ and θ, and introducing k^2, for simplicity, we have

$$s = C\frac{k^2}{g}\log_e\frac{(i)-(\theta)}{(i)-(\phi)};$$

or, if we use common logarithms,

$$s = CM\frac{k^2}{g}\log\frac{(i)-(\theta)}{(i)-(\phi)}, \quad . \quad . \quad . \quad . \quad . \quad (31')$$

in which $M = 2.30259$.

The value of k^2 (or, rather, its logarithm) as deduced by Mayevski from the Krupp experiments with projectiles having ogival heads struck with radii of two calibres, and reduced to English units, is

$$\log k^2 = 5.53676;$$

and, therefore,

$$\log M\frac{k^2}{g} = 4.39131.$$

The above equation gives the length of the trajectory from the origin to any point whose inclination is θ. The expression for the velocity at the same point is (Eq. (17')),

$$v_\theta = \sqrt{C}\frac{k\sec\theta}{\sqrt{(i)-(\theta)}}. \quad . \quad . \quad . \quad . \quad . \quad (32')$$

Example 1. The 12-in. M. L. rifled mortar fires a projectile weighing 610 lbs.

If the M. V. is 800 f. s., and angle of departure 60°, what will be the distance travelled by the projectile on returning to the level of the gun, and the striking velocity?

Here $d = 12$ in., $w = 610$ lbs., $c = 1$, $\log\ C = 0.62697$, $V = 800$ f. s., $\phi = 60°$, and $\theta = -63°\ 17' = -\omega$.

From Table IV we take $(\phi) = 2.39053$ and $(\theta) = -2.92859$. The computations are now as follows:

$$\begin{aligned} \log k^2 &= 5.53676 \\ \log C &= 0.62697 \\ \hline \log Ck^2 &= 6.16373 \end{aligned}$$

$$
\begin{aligned}
2 \log V &= 5.80618 \\
2 \log \cos \phi &= 9.39794 \\
\hline
\log V_1^2 &= 5.20412 \\
\hline
\end{aligned}
$$

$$
\begin{aligned}
\log\ 9.11200 &= 0.95961 = \log [(i) - (\phi)] \\
(\phi) &= \ 2.39053 \\
\hline
(i) &= 11.50253 \\
(\theta) &= -\ \ 2.92859 \\
\hline
\log 14.43112 &= 1.15930 = \log [(i) - (\theta)]
\end{aligned}
$$

$$
\begin{aligned}
\log 0.19969 &= 9.30036 \\
\log M \frac{k^2}{g} &= 4\ 39131 \\
\log C &= 0.62697 \\
\hline
\log s &= 4.31864 \\
\therefore s &= 20828 \text{ feet.}
\end{aligned}
$$

$$
\begin{aligned}
\log \sqrt{C}k &= 3.08186 \\
\log \sec \theta &= 0.34719 \\
\hline
& \ 3.42905 \\
\tfrac{1}{2} \log [(i) - (\theta)] &= 0.57965 \\
\hline
\log v_\theta &= 2.84940 \qquad \therefore v_\theta = 707.0 \text{ f. s.}
\end{aligned}
$$

Expressions for the Co-ordinates x and y.—Equation (31′) gives the value of s reckoned from the origin to any given point, and is analytically exact for the assumed law of resistance.

To make use of this equation for computing the co-ordinates of the given point we proceed as follows: If s' is the length of an arc of the trajectory from the origin to where the inclination is θ', and s'' the length to some other point further on where the inclination is θ'' ($\theta' > \theta''$), we shall have from (31′)

$$
s' = CM\frac{k^2}{g} \log \frac{(i) - (\theta')}{(i) - (\phi)},
$$

and

$$s'' = CM\frac{k^2}{g}\log\frac{(i) - (\theta'')}{(i) - (\phi)};$$

whence, subtracting the first equation from the second, member by member, thereby eliminating the factor $[(i) - (\phi)]$, we have

$$s'' - s' \doteq \Delta s = CM\frac{k^2}{g}\log\frac{(i) - (\theta'')}{(i) - (\theta')},$$

an equation which gives the length of any portion of a trajectory in terms of the inclinations of its extremities.

If θ'' differs but little from θ' (say one degree), Δs will be very nearly a straight line having a mean inclination of $\frac{1}{2}(\theta' + \theta'')$, and will be the hypothenuse of a right-angled triangle whose base is Δx and altitude Δy. We therefore have

$$\Delta x = CM\frac{k^2}{g}\log\frac{(i) - (\theta'')}{(i) - (\theta')}\cos\tfrac{1}{2}(\theta' + \theta''),$$

$$\Delta y = CM\frac{k^2}{g}\log\frac{(i) - (\theta'')}{(i) - (\theta')}\sin\tfrac{1}{2}(\theta' + \theta'').$$

Making

$$\Delta\xi = \log\frac{(i) - (\theta'')}{(i) - (\theta')}\cos\tfrac{1}{2}(\theta' + \theta''),$$

and

$$\Delta\zeta = \log\frac{(i) - (\theta'')}{(i) - (\theta')}\sin\tfrac{1}{2}(\theta' + \theta''),$$

we have

$$\Delta x = CM\frac{k^2}{g}\Delta\xi;$$

$$\Delta y = CM\frac{k^2}{g}\Delta\zeta.$$

For the horizontal range we evidently have

$$X = \Sigma \Delta x = CM\frac{k^2}{g}\Sigma\Delta\xi = CM\frac{k^2}{g}\xi,$$

the summation extending from $\theta = \phi$ to $\theta = \omega$, or the angle of fall.

To determine the maximum ordinate, the summation $\Sigma\zeta$ is taken from $\theta = \phi$ to $\theta = 0$, and is continued in the descending branch to ω (the angle of fall), determined by the condition

$$\Sigma\Delta\zeta = 0,$$

since the sum of the negative increments of y in the descending branch is equal numerically to the sum of the positive increments in the ascending branch. We therefore have

$$y_0 = CM\frac{k^2}{g}\zeta_0.$$

Example 2. Compute the values of Δx and Δy with the data of Ex. 1, for the arc comprised between $\theta' = 25°$ and $\theta'' = 24°$.

We have from Ex. 1, $(i) = 11.50253$; and from Table IV, $(\theta') = 0.48269$ and $(\theta'') = 0.45953$. We also have

$$\tfrac{1}{2}(\theta' + \theta'') = 24° \ 30';$$

whence

$$\Delta\xi = \log\frac{11.50253 - 0.45953}{11.50253 - 0.48269} \times \cos 24° \ 30'$$

$$= 0.00082970;$$

$$\Delta\zeta = \log\frac{11.50253 - 0.45953}{11.50253 - 0.48269} \times \sin 24° \ 30'$$

$$= 0.00037812.$$

Multiplying $CM\frac{k^2}{g}$ by these numbers, we have

$$\Delta x = 86.54 \text{ feet};$$

$$\Delta y = 39.44 \quad \text{"}$$

also

$$\Delta s = 95.10 \quad \text{"}$$

Expression for the Time.—We have for the time of describing any small portion of the trajectory, the expression

$$\Delta t = \frac{\Delta x}{v_1},$$

in which v_1 is the mean horizontal velocity corresponding to Δx; but from (32′) we have

$$v_1 = \frac{\sqrt{C}k}{\sqrt{(i) - (\theta)}};$$

whence

$$\Delta t = \frac{\Delta x \sqrt{(i) - (\theta)}}{\sqrt{C}k};$$

or, substituting for Δx its value already given,

$$\Delta t = \sqrt{C}\,\frac{Mk}{g}\,\Delta\xi\,\sqrt{(i) - (\theta)}.$$

If we put

$$\Delta\Theta = \Delta\xi\,\sqrt{(i) - (\theta)},$$

the expression for Δt becomes

$$\Delta t = \sqrt{C}\frac{Mk}{g}\Delta\Theta.$$

We may compute $\Delta\Theta$ with great accuracy as follows: Taking logarithms, we have

$$\log \Delta\Theta = \log \Delta\xi + \tfrac{1}{2} \log [(i) - (\theta)].$$

The two values of log $[(i) - (\theta)]$ corresponding to the extremities of Δs, are log $[(i) - (\theta')]$ and log $[(i) - (\theta'')]$, the first of which is too small, and the second too great; whence, taking their arithmetical mean,

$$\log \Delta\Theta = \log \Delta\xi + \tfrac{1}{4} \log [(i) - (\theta')] + \tfrac{1}{4} \log [(i) - (\theta'')],$$

by means of which Θ can be computed. We then have

$$T = \sqrt{C} M \frac{k}{g} \Theta,$$

the summation extending the same as in determining the range. The logarithm of $M\frac{k}{g}$ is 1.62293.

Tables.—General Otto of the Prussian Artillery published several years ago extensive tables of ξ, ζ, and Θ*, for values of ϕ beginning at 30° and continuing by intervals of 5° up to 75°. The argument for each of these tables is i, which must be first computed from the given data by the equation.

$$(i) = C\frac{k^2}{V_1^2} + (\phi),$$

and then entering the proper table with this value of i, the values of ξ, ζ, and Θ are immediately found, and from which X, y_0, and T can be easily computed by the formulæ already given. We give the following example illustrating this method. The tables referred to are those in Otto's work.

Example 3. Compute the range, time of flight, angle of fall, and maximum ordinate of the trajectory described by the shot of Ex. 1.

We found (i) in Ex. 1 to be 11.50253; and therefore (Table 1) $i = 77° \; 29'.18$. Next, from Table 2, for $\phi = 60°$, and with the argument $i = 77° \; 29'.18$, we find $\xi = 0.1402$ and

---

* "Taffeln für den Bombenwurf." Translated into French by Rieffel with the title "Tables Balistiques Générales pour le Tir Élevé." Paris, 1844.

$\Theta = 0.4750$. From Table 3, with the same arguments, we get $\omega = 63° 17'$; and from Table 4, $\zeta_0 = 0.0652$.

$$\log \xi = 9.14675$$
$$\log M\frac{k^2}{g} = 4.39131$$
$$\log C = 0.62697$$
$$\log X = 4.16503 \qquad \therefore X = 14623 \text{ feet.}$$

$$\log \Theta = 9.67669$$
$$\log M\frac{k}{g} = 1.62293$$
$$\tfrac{1}{2}\log C = 0.31348$$
$$\log T = 1.61310 \qquad \therefore T = 41''.03.$$

$$\log \zeta_0 = 8.81425$$
$$\log CM\frac{k^2}{g} = 5.01828$$
$$\log y_0 = 3.83253 \qquad \therefore y_0 = 6800 \text{ feet.}$$

Modification of Euler's Method and Otto's Tables.— Instead of taking i for the argument, we may shorten the calculations and greatly abridge the tables by taking $\frac{V}{\sqrt{C}}$ for the argument. The new tables will be constructed as follows: Let

$$\frac{V}{\sqrt{C}} = V_0;$$

then the first expression for (i) on page 285 becomes, by a slight reduction,

$$(i) = (\phi) + \frac{k^2}{V_0^2 \cos^2 \phi},$$

by means of which i can be computed for given values of ϕ and V_0. Now with the given value of ϕ and the computed

value of i enter Otto's tables, and take out ξ, Θ, and ω. We then have, by obvious modifications of the formulæ already given,

$$\frac{X}{C} = M\frac{k^2}{g}\xi;$$

$$\frac{T}{\sqrt{C}} = M\frac{k}{g}\Theta;$$

$$\frac{v_\omega}{\sqrt{C}} = \frac{k \sec \omega}{\sqrt{(i)+(\omega)}}.$$

Table V gives the values of the first members of the above equations (and also ω) for values of $\frac{V}{\sqrt{C}}$ extending from 300 to 500, with a common difference of 10.

These limits are extensive enough for the solution of most of the problems of mortar firing, when the muzzle velocity does not exceed 800 f. s. The table would be more complete, however, if $\frac{V}{\sqrt{C}}$ extended from 200 to 600.

The expression for the height of the summit is given at the top of each page.

Example 4. Given $V = 206.6$ m. s. $= 677.8$ f. s., $d = 21$ cm., $w = 91$ kg., and $\phi = 60°$, to compute the horizontal range, time of flight, angle of fall, and striking velocity.

To compute log C, expressed in English units, when w is given in kilogrammes and d in centimetres (c and $\frac{\delta_1}{\delta}$ both being unity), we make use of the equation

$$\log C = 1.15298 + \log w - 2 \log d.$$

$$\begin{aligned} \text{const. log} &= 1.15298 \\ \log w &= 1.95904 \\ \text{a. c. } 2 \log d &= 7.35556 \\ \hline \log C &= 0.46758 \end{aligned}$$

$$\log V = 2.83110$$
$$\tfrac{1}{2}\log C = 0.23379$$
$$\log 395.6 = 2.59731$$

Next, from Table V, $\phi = 60°$, we find, for the argument 395.6,

$$\frac{X}{C} = 3472 + .56 \times 153 = 3557.7;$$

$$\frac{T}{\sqrt{C}} = 20.00 + .56 \times .46 = 20.26;$$

$$\omega = 63° \; 18' + .56 \times 8' = 63° \; 22';$$

$$\frac{v_\omega}{\sqrt{C}} = 344 + .56 \times 7 = 348.$$

We now easily find

$$X = 10441 \text{ feet};$$
$$T = 34.71;$$
$$\omega = 63° \; 22';$$
$$v_\omega = 596 \text{ f. s.};$$
$$y_0 = 0.47X = 4900 \text{ feet}.$$

The mean observed range was 10385 feet. The range of one of the shots was 10440 feet.

Example 5. Given $V = 204.1$ m. s., $= 669.6$ f. s., $d = 21$ cm., $w = 91$ kg., and $\phi = 45°$, to compute X, T, ω, and v_ω.

Proceeding as in Ex. 4, we get

$$X = 11922 \text{ feet};$$
$$T = 28''.16;$$
$$\omega = 48° \; 58';$$
$$v_\omega = 578 \text{ f. s.};$$
$$y_0 = 0.27X = 3200 \text{ feet}.$$

The observed ranges varied considerably among themselves. One of them was 11923 feet and another 11920 feet; while still another was 11749 feet.

Example 6. Data the same as in Ex. 5, except that $\phi = 30°$. We find

$$X = 10659 \text{ feet};$$
$$T = 20''.13;$$
$$\omega = 33° \, 02';$$
$$v_\omega = 578 \text{ f. s.};$$
$$y_0 = 0.15X = 1600 \text{ feet}.$$

Example 7. The 10-inch rifled mortar No. 2 was fired at Sandy Hook, January 4, 1884, at 60° elevation, weight of projectile 348 lbs., M. V. 660 f. s. Required the range, time of flight, angle of fall, and striking velocity.

Here $w = 348$, $d = 10$, $V = 660$, and $\phi = 60°$. The barometer stood at 30.293 in. and thermometer at 30°; therefore $\frac{\delta_1}{\delta} = 0.932$.

As Table V was computed for projectiles having ogival heads struck with a radius of 2 calibres, we must make in this case $c = \frac{10}{9}$. From this data we find $\log C = 0.46524$ and $\frac{V}{\sqrt{C}} = 386.3$.

Next from Table V with the given argument we easily find

$$\frac{X}{C} = 3416; \quad \frac{T}{\sqrt{C}} = 19.83; \quad \omega = 63° \, 15'; \quad \frac{v_\omega}{\sqrt{C}} = 341;$$

and therefore

$$X = 3324 \text{ yards};$$
$$T = 33''.88;$$
$$v_\omega = 582.6.$$

The observed range of this shot was 3322 yards.

Example 8. With the data of Ex. 7, what should the M.V. be for a range of 4000 yards?

We have

$$\begin{aligned} \log X &= 4.07918 \\ \log C &= 0.46524 \\ \hline \log 4111 &= 3.61394 \end{aligned}$$

With this value of $\frac{X}{C}$ we find from Table V,

$$\frac{V}{\sqrt{C}} = 431.3;$$

$$\therefore V = 737 \text{ f. s.}$$

Having computed the required muzzle velocity, the amount of powder necessary can be determined by the formula on page 164.

This method is the simplest and most accurate of any yet proposed when the muzzle velocity does not exceed 800 f. s. It labors under the disadvantage, however, of not being directly applicable to all angles of elevation employed in mortar firing.

Siacci's Method for Curved and High-Angle Fire.—*Siacci* has recently published* a new method for the solution of problems in curved and high-angle fire. He has, in fact, utilized his equations for direct fire for this purpose, namely:

$$X = C\{S(u) - S(V)\};$$

and

$$\sin 2\phi = C\left\{\frac{A(u) - A(V)}{S(u) - S(V)} - I(V)\right\}.$$

He does this by means of an integrating factor, β, called by Captain *Vallier* of the French Artillery, the "parameter of curvature."†

* See *Rivista d'Artiglieria e Genio*, for December, 1889.

† *Revue d'Artillerie*, vol. xxxvi, p. 153.

The nature of this factor and the method of introducing it into the above equations are easily understood. Take the differential equation (Appendix I, p. 270)

$$dx = -\frac{C}{A \sec^{n-1}\theta}\frac{dv_1}{v_1^{n-1}}.$$

In deducing the equation for the horizontal range (X) from this, we assumed

$$\sec^{n-1}\theta = \sec^{n-2}\phi,$$

which, as has been shown, gives good results for direct fire, on account of the small value of θ. In the article cited above, Siacci makes

$$\sec^{n-1}\theta = \beta \sec^{n-2}\phi,$$

and then determines the values of β which satisfy both the above equations for X and ϕ, for given values of these quantities. Substituting for $\sec^{n-1}\theta$ the value given above, the expression for dx becomes

$$dx = -\frac{C}{A\beta}\frac{du}{u^{n-1}},$$

in which

$$u = v_1 \sec\phi = v\frac{\cos\theta}{\cos\phi}.$$

As β is constant for a given value of X, corresponding to a given value of ϕ, we may introduce it into the ballistic coefficient C, which thus becomes

$$C = \frac{\delta_1}{\delta}\cdot\frac{w}{cd^2\beta}.$$

Table VI gives the values of β calculated for all ranges which satisfy the condition

$$X < 12000 \sin 2\phi,$$

in which X is in metres.

Example 9. Compute the trajectory of Ex. 4 by Siacci's method.

The range in this example is in the neighborhood of 3200 m. We therefore find from Table VI, for $\phi = 60°$ and $X = 3200$ m., $\beta = 1.425$. Also, as the elements are to be computed by means of Table I, we must make $c = 0.9$. Therefore in this case we have for computing C the equation

$$\log C = 1.19874 + \log w - 2 \log d - \log \beta.$$

$$\begin{aligned} \text{const. } \log &= 1.19874 \quad \text{(See Second Method,} \\ \log w &= 1.95904 \quad \text{Prob. IX, p. 61.)} \\ \text{a. c. } 2 \log d &= 7.35556 \\ \text{a. c. } \log \beta &= 9.84619 \\ \hline \log C &= 0.35953 \\ \log \sin 2\phi &= 9.93753 \\ \hline \log 0.37844 &= 9.57800 \end{aligned}$$

$$\begin{aligned} I(V) &= 0.83129 \\ \hline I(u_0) &= 1.20973 \end{aligned}$$

We therefore have the equation

$$\frac{A(u) - 3589.3}{S(u) - 13857.9} = 1.20973,$$

which we find by a few trials is satisfied when $u = 520$. The remaining computations are as follows:

$$\begin{aligned} S(u) &= 18354.7 \\ S(V) &= 13857.9 \\ \hline \log 4496.8 &= 3.65290 \\ \log C &= 0.35953 \\ \hline \log X &= 4.01243 \qquad \therefore X = 10290 \text{ ft.} \end{aligned}$$

$$
\begin{aligned}
I(u) &= 1.66155 \\
I(u_0) &= 1.20973 \\
\hline
\log 0.45182 &= 9.65497 \\
\log C &= 0.35953 \\
\log 0.5 &= 9.69897 \\
\text{a. c. } 2 \log \cos \phi &= 0.60206 \\
\hline
\log \tan \omega &= 0.31553 \qquad \therefore \omega = 64^\circ\ 12' \\
\\
\log u &= 2.71600 \\
\log \cos \phi &= 9.69897 \\
\text{a. c. } \log \cos \omega &= 0.36128 \\
\hline
\log v &= 2.77625 \qquad \therefore v = 597 \text{ f. s.} \\
\\
T(u) &= 20.130 \\
T(V) &= 12.533 \\
\hline
\log 7.597 &= 0.88064 \\
\log C &= 0.35953 \\
\text{a. c. } \log \cos \phi &= 0.30103 \\
\hline
\log T &= 1.54120 \qquad \therefore T = 34''.77.
\end{aligned}
$$

These results are very close approximations. But for velocities exceeding 800 f. s., and with small values of C, the ranges computed by means of the factor β are considerably too great, as was pointed out by Captain Vallier in the article already cited.

Didion's Method for High-angle Fire.—Equations (15′), (7′), (8′), and (10′) may be written, respectively,

$$\frac{d\theta}{\cos^2 \theta} = \frac{gC}{A} \frac{\sec^2 \theta dv_1}{(v_1 \sec \theta)^{n+1}};$$

$$dt = -\frac{C}{A} \frac{\sec \theta dv_1}{(v_1 \sec \theta)^n};$$

$$dx = -\frac{C}{A}\frac{dv_1}{(v_1 \sec\theta)^{n-1}};$$

$$ds = -\frac{C}{A}\frac{\sec\theta dv_1}{(v_1 \sec\theta)^{n-1}}.$$

The second members of these equations cannot generally be integrated, since the relations between v_1 and $\sec\theta$, given by the integration of (15′), renders them quite intractable. It is evident, however, that if $\sec\theta$ were constant they could be integrated immediately by the elementary rules of integration; and, also, that $\sec\theta$ may be replaced by some mean value (not necessarily the same in all the equations) between the limits of integration.

Let α be a mean value of $\sec\theta$ between the limits of $\sec\phi$ and $\sec\theta$, which will satisfy the first of the above equations, after integration. We then have

$$\frac{d\theta}{\cos^2\theta} = \frac{\alpha g C}{A}\frac{d(\alpha v_1)}{(\alpha v_1)^{n+1}}.$$

Writing u for $\alpha v_1 = \alpha v \cos\theta$, and U for $\alpha V_1 = \alpha V \cos\phi$, and integrating between those limits, we have

$$\tan\phi - \tan\theta = \frac{\alpha g C}{A}\int_u^U \frac{du}{u^{n+1}}.$$

Comparing this with the corresponding equation in Appendix I (page 271), it will be seen that we have

$$\tan\phi - \tan\theta = \frac{\alpha C}{2}\left\{ I(u) - I(U) \right\}.* \quad . \quad . \quad (33')$$

The only ambiguity in this equation is in the value to be assigned to α. If we compare (33′) with (16′) (which latter is analytically exact), we shall find that in this case

$$\alpha = \left\{ \frac{(\phi) - (\theta)}{\tan\phi - \tan\theta} \right\}^{\frac{1}{n-1}};$$

* Note the difference between u as used here and in Appendix I.

in which

$$(\theta) = \int \frac{d\theta}{\cos^{n+1}\theta}.$$

And similarly for (ϕ).

When $n = 2$, we have

$$\alpha = \frac{(\phi) - (\theta)}{\tan\phi - \tan\theta},$$

in which

$$(\theta) = \int \frac{d\theta}{\cos^3\theta} = \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\log_\epsilon(\tan\theta + \sec\theta).$$

This last value of α renders (33′) analytically exact when the law of resistance is that of the square of the velocity.

We will next consider the expression for ds. Calling, as before, α some mean value of $\sec\theta$ between the limits of integration, which satisfies the equation, we have

$$ds = -\frac{C}{A}\cdot\frac{d(\alpha v_1)}{(\alpha v_1)^{n-1}} = -\frac{C}{A}\frac{du}{u^{n-1}};$$

whence, integrating between the limits U and u, and o and s, we have, finally,

$$S = C\left\{S(u) - S(U)\right\}. \quad \text{. . . . . . .} \quad (34')$$

Comparing (34′) with (30′) (which latter is analytically exact when $n = 2$), it will be seen that α may have *any positive value* in (34′) when $n = 2$, and still give correct values to s. Therefore the value

$$\alpha = \frac{(\phi) - (\theta)}{\tan\phi - \tan\theta},$$

in which

$$(\theta) = \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\log_\epsilon(\tan\theta + \sec\theta)$$

renders both (33′) and (34′) analytically exact when the law of resistance is that of the square of the velocity. We may there-

fore fairly assume that the same value of α in the expressions for dx and dt will furnish results which are close approximations to the truth, when the same law of resistance holds.

Proceeding in the same way with these last equations, we obtain without any difficulty

$$t = C\{T(u) - T(U)\}; \quad . \quad . \quad . \quad . \quad . \quad (35')$$

$$x = \frac{C}{\alpha}\left\{S(u) - S(U)\right\}. \quad . \quad . \quad . \quad . \quad (36')$$

Finally, operating in the same manner for $\frac{y}{x}$ as in Appendix I, we find

$$\frac{y}{x} = \tan\phi - \frac{\alpha C}{2}\left\{\frac{A(u) - A(U)}{S(u) - S(U)} - I(U)\right\}. \quad . \quad (37')$$

In all these equations

$$\left.\begin{array}{l} U = \alpha V \cos\phi, \\ u = \alpha v \cos\theta. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38')$$

The values of the functions (ϕ) and (θ) are given in Table IV at the end of this work. The same table also gives the natural tangents to be used in connection with the functions.

We may also deduce an approximate value for α by the following considerations: Since

$$\sec\theta = \frac{ds}{dx},$$

that is, the ratio of an element of the trajectory to its horizontal projection, it is evident that a suitable value for α would be

$$\alpha = \frac{s}{x},$$

which is manifestly a mean of all the values of $\frac{ds}{dx}$ between the given limits. As, however, this ratio cannot be found, General Didion, in his classic work *Traité de Balistique*, assumed for α

the ratio of a parabolic arc (or trajectory *in vacuo*), whose extremities have the same inclinations as the arc of the trajectory in question, to its horizontal projection.

To determine this ratio we have only to divide (25′) by (23′), which gives

$$\frac{s}{x} = \alpha = \frac{(\phi) - (\theta)}{\tan \phi - \tan \theta},$$

which is the same expression for α as before determined by other considerations.

We have shown that the adopted value for α gives the *true velocity* of the projectile at any point of the trajectory; and, also, the *exact length of the curve* described from the origin, when the law of resistance is that of the square of the velocity; and we have thence assumed that it will also give sufficiently accurate values for the horizontal distance passed over, and for the time of flight, *for the same law of resistance.* To verify this assumption, and also to show the degree of approximation attained when the law of resistance is that of the cube or of the sixth power of the velocity, we must have recourse to quadratures for calculating the exact values of the definite integrals involved.

Example 10.—Let $\phi = 30°$, $\theta = 24°$, and $C = \frac{800}{144}$. Compute v, t, x, y, and s when $V = 1886$ f. s. by quadratures, and also by means of α.

The following are the results:

$$\rho \propto v^2.$$

	By Quadratures.	By means of α.	Difference.
v	1400.4 f. s.	1400.4 f. s.	0
t	5″.888	5″.894	− 0″.006
x	8481.4 ft.	8499.9 ft.	− 18.5 ft.
y	4381.9 ft.	4392.7 ft.	− 10.8 ft.
s	9550.6 ft.	9550.6 ft.	0

Of these elements v and s are, of course, the same by both methods of calculation; while t, x, and y are too great when

computed by means of α. The error is, however, less than one-fourth of one per cent.

Let $V = 1330$ f. s., ϕ and θ remaining as before.

$$\rho \propto v^3.$$

	By Quadratures.	Using α.	Difference.
v	1110 f. s.	1110 f. s.	0
x	4761.6 ft.	4773.7	− 12.1 ft.

Let $v = 1120$ f. s., ϕ and θ the same as before.

$$\rho \propto v^6.$$

	By Quadratures.	Using α.	Difference.
v	991.2 f. s.	991.9	+ 0.7
x	3583.1 ft.	3558.4 ft.	+ 24.7 ft.

It will be seen from the above that the values of x given by α are slightly too great when $n = 3$, and too small when $n = 6$; while the velocity is practically correct in both cases. We will now compute a complete trajectory and compare our results with actual experiment.

In applying Didion's value of α to the computation of a complete trajectory *at one operation*, having given the muzzle velocity and angle of departure, we should compute α by the equation

$$\alpha = \frac{(\phi) + (\omega)}{\tan \phi + \tan \omega},$$

ω being the angle of fall, and considered positive. As the angle of fall is not known, we will suppose it equal to the angle of departure; whence we have

$$\alpha = \frac{(\phi)}{\tan \phi}.$$

Example 11. Compute the trajectory of the Jubilee-shots fired at Shoeburyness in April, 1888.

The data for this shot, as communicated to the author by Prof. A. G. Greenhill, March 9, 1888, are as follows:

$$w = 380 \text{ lbs.}, \quad d = 9.15 \text{ in.}, \quad \phi = 40^\circ, \quad \text{and} \quad V = 2360 \text{ f. s.}$$

The calculations, which were made before the shots were fired, are as follows:

From Table IV, we find $(\phi) = 0.92914$

$$\begin{aligned} \log (\phi) &= 9.96808 \\ \log \tan \phi &= 9.92381 \\ \hline \log \alpha &= 0.04427 \\ \log V &= 3.37291 \\ \log \cos \phi &= 9.88425 \\ \hline \log U &= 3.30143 \quad \therefore \quad U = 2001.86 \end{aligned}$$

We will next compute the ballistic coefficient for the level of the sea, which we will designate by C', to be corrected later for altitude. The coefficient of reduction (c) we will take equal to 0.914, which is its theoretical value.

$$\begin{aligned} \log w &= 2.57978 \\ \text{a. c. } \log d^2 &= 8.07716 \\ \text{a. c. } \log c &= 0.03905 \\ \hline \log C' &= 0.69599 \end{aligned}$$

The next step is to compute the height of summit in order to get a correction for the ballistic coefficient due to the decrease of density of the air. From (31′) we obtain, when $\theta = 0$,

$$I(u_0) = \frac{2 \tan \phi}{\alpha C} + I(U). \quad . \quad . \quad . \quad . \quad (37')$$

Also, eliminating $\tan \phi$ from (35′) and (37′), and making

$$z_0 = S(u_0) - S(U),$$

we have at the summit

$$y_0 = \frac{C^2}{2}\left\{ I(u_0)z_0 + A(U) - A(u_0) \right\} . \quad . \quad . \quad (38')$$

$$\begin{aligned}
\log 2 &= 0.30103 \\
\log \tan \phi &= 9.92381 \\
\text{a. c. } \log \alpha &= 9.95573 \\
\text{a. c. } \log C' &= 9.30401 \\
\hline
\log 0.30520 &= 9.48458
\end{aligned}$$

$$\begin{aligned}
I(U) &= 0.02761 \\
\hline
I(u_0) &= 0.33281 \quad \therefore \; u_0 = 898.15
\end{aligned}$$

$$\begin{aligned}
S(u_0) &= 9228.5 \\
S(U) &= 2361.7 \\
\hline
\log 6866.8 &= 3.83675 \\
\log I(u_0) &= 9.52220 \\
\hline
\log 2285.32 &= 3.35895 \\
A(U) &= 28.98 \\
-A(u) &= -1001.17 \\
\hline
\log 1313.13 &= 3.11831 \\
2 \log C &= 1.39198 \\
\text{a. c. } \log 2 &= 9.69897 \\
\hline
\log y_0 &= 4.20926 \quad \therefore \; y_0 = 16191 \text{ ft.}
\end{aligned}$$

Following the rule given on page 88, we find

$$h = \frac{2}{3} \times 16191 = 10794 \text{ ft.}$$

As this is beyond the limit of the table on page 88, it will be necessary to compute the altitude factor, or rather its

logarithm, which can easily be done as follows: Designating it by f, we have

$$f = e^{\frac{h}{\lambda}};$$

$$\therefore \log(\log f) = \log(\log e) - \log\lambda + \log h.$$

The value of $\log(\log e) - \log\lambda$ is $5.19374 - 10$; and therefore

$$\log(\log f) = \log h - 4.80626.$$

In our example we have

$$\begin{aligned}
\log h &= 4.03318 \\
\text{const. } \log &= 4.80626 \\
\hline
\log(\log f) &= 9.22692 \\
\therefore \log f &\doteq 0.16862 \\
\log C' &= 0.69599 \\
\hline
\log C &= 0.86461 \\
\log \frac{2\tan\phi}{\alpha} &= 0.18057 \\
\hline
\log 0.20700 &= 9.31596
\end{aligned}$$

$$\begin{aligned}
I(U) &= 0.02761 \\
\hline
I(u_0) &= 0.23461 \quad \therefore u_0 = 988.1
\end{aligned}$$

We have from (35′) when $y = 0$,

$$\frac{A(u_\omega) - A(U)}{S(u_\omega) - S(U)} = \frac{2\tan\phi}{\alpha C} + I(U) = I(u_0).$$

We therefore have the equation

$$\frac{A(u_\omega) - 28.98}{S(u_\omega) - 2361.7} = 0.23461,$$

from which to find u_ω by trial, as explained in the second method, Prob. IX. We find by a few trials that

$$u_\omega = 753.8.$$

We are now prepared to compute the range as follows:

$$
\begin{aligned}
S(u_\omega) &= 12054.7 \\
S(U) &= 2361.7 \\
\hline
\log\ 9693.0 &= 3.98646 \\
\log C &= 0.86461 \\
\text{a. c.} \log \alpha &= 9.95573 \\
\hline
\log X &= 4.80680 \\
\therefore X &= 64091 \text{ feet} \\
&= 21364 \text{ yards.}
\end{aligned}
$$

The ranges of the two shots fired April 15, 1888, were 21048 yards and 21358 yards, respectively.

Mean observed range,	21203	yards.
Computed range,	21364	"
Difference,	161	"

The time of flight is computed as follows:

$$
\begin{aligned}
T(u) &= 10.009 \\
T(U) &= 1.002 \\
\hline
\log\ 9.007 &= 0.95458 \\
\log C &= 0.86461 \\
\hline
\log T &= 1.81919 \\
\therefore T &= 65.9 \text{ seconds,}
\end{aligned}
$$

which is very nearly correct.

TABLES.

TABLE I.

Ballistic Table for Ogival-Headed Projectiles.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
2800	000.0	1268	0.00	7	0.00000	106	0.000	46	0.000	139	0.00107	6
2750	126.8	1292	0.07	21	0.00106	112	0.046	47	0.139	150	0.00113	6
2700	256.0	1315	0.28	36	0.00218	118	0.093	49	0.289	161	0.00119	7
2650	387.5	1341	0.64	54	0.00336	125	0.142	51	0.450	174	0.00126	7
2600	521.6	1367	1.18	71	0.00461	133	0.193	53	0.624	188	0.00133	8
2550	658.3	1393	1.89	93	0.00594	140	0.246	56	0.812	203	0.00141	9
2500	797.6	1422	2.82	115	0.00734	149	0.302	57	1.015	220	0.00150	10
2450	939.8	1452	3.97	140	0.00883	160	0.359	60	1.235	239	0.00160	10
2400	1085.0	1481	5.37	166	0.01043	169	0.419	62	1.474	260	0.00170	11
2350	1233.1	1514	7.03	197	0.01212	180	0.481	65	1.734	283	0.00181	12
2300	1384.5	1547	9.00	231	0.01392	192	0.546	68	2.017	308	0.00193	13
2250	1539.2	1582	11.31	266	0.01584	205	0.614	72	2.325	337	0.00206	14
2200	1697.4	321	13.97	58	0.01789	43	0.686	14	2.662	71	0.00220	3
2190	1729.5	322	14.55	60	0.01832	44	0.700	15	2.733	72	0.00223	3
2180	1761.7	323	15.15	62	0.01876	44	0.715	15	2.805	74	0.00226	4
2170	1794.0	325	15.77	63	0.01920	44	0.730	15	2.879	75	0.00230	3
2160	1826.5	327	16.40	65	0.01964	46	0.745	15	2.954	77	0.00233	3
2150	1859.2	328	17.05	67	0.02010	46	0.760	15	3.031	78	0.00236	3
2140	1892.0	329	17.72	68	0.02056	46	0.775	16	3.109	79	0.00239	4
2130	1924.9	331	18.40	70	0.02102	47	0.791	15	3.188	81	0.00243	3
2120	1958.0	333	19.10	73	0.02149	48	0.806	16	3.269	83	0.00246	4
2110	1991.3	335	19.83	74	0.02197	49	0.822	16	3.352	84	0.00250	3
2100	2024.8	336	20.57	76	0.02246	49	0.838	16	3.436	86	0.00253	4
2090	2058.4	337	21.33	79	0.02295	50	0.854	16	3.522	87	0.00257	4
2080	2092.1	339	22.12	80	0.02345	51	0.870	16	3.609	89	0.00261	4
2070	2126.0	341	22.92	82	0.02396	51	0.886	17	3.698	91	0.00265	3
2060	2160.1	343	23.74	85	0.02447	52	0.903	17	3.789	93	0.00268	4
2050	2194.4	344	24.59	87	0.02499	53	0.920	17	3.882	94	0.00272	4
2040	2228.8	346	25.46	89	0.02552	54	0.937	17	3.976	96	0.00276	4
2030	2263.4	348	26.35	91	0.02606	54	0.954	17	4.072	99	0.00280	5

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
2020	2298.2	349	27.26	94	0.02660	55	0.971	17	4.171	100	0.00285	4
2010	2333.1	351	28.20	96	0.02715	57	0.988	17	4.271	102	0.00289	4
2000	2368.2	353	29.16	98	0.02772	57	1.005	18	4.373	104	0.00293	5
1990	2403.5	355	30.14	101	0.02829	57	1.023	18	4.477	107	0.00298	4
1980	2439.0	356	31.15	104	0.02886	59	1.041	18	4.584	108	0.00302	5
1970	2474.6	358	32.19	107	0.02945	60	1.059	18	4.692	111	0.00307	5
1960	2510.4	360	33.26	109	0.03005	61	1.077	19	4.803	113	0.00312	4
1950	2546.4	362	34.35	113	0.03066	61	1.096	18	4.916	115	0.00316	5
1940	2582.6	363	35.48	115	0.03127	62	1.114	19	5.031	118	0.00321	5
1930	2618.9	366	36.63	118	0.03189	64	1.133	19	5.149	120	0.00326	5
1920	2655.5	367	37.81	121	0.03253	65	1.152	19	5.269	123	0.00331	6
1910	2692.2	370	39.02	124	0.03318	65	1.171	20	5.392	126	0.00337	5
1900	2729.2	371	40.26	127	0.03383	67	1.191	19	5.518	128	0.00342	6
1890	2766.3	374	41.53	130	0.03450	67	1.210	20	5.646	130	0.00348	5
1880	2803.7	375	42.83	133	0.03517	69	1.230	20	5.776	134	0.00353	6
1870	2841.2	377	44.16	137	0.03586	70	1.250	20	5.910	136	0.00359	6
1860	2878.9	380	45.53	140	0.03656	71	1.270	21	6.046	140	0.00365	6
1850	2916.9	382	46.93	143	0.03727	72	1.291	20	6.186	142	0.00371	6
1840	2955.1	383	48.36	147	0.03799	73	1.311	21	6.328	146	0.00377	6
1830	2993.4	386	49.83	151	0.03872	74	1.332	21	6.474	149	0.00383	6
1820	3032.0	388	51.34	155	0.03946	76	1.353	22	6.623	152	0.00389	7
1810	3070.8	390	52.89	158	0.04022	77	1.375	21	6.775	156	0.00396	6
1800	3109.8	392	54.47	162	0.04099	78	1.396	22	6.931	159	0.00402	7
1790	3149.0	394	56.09	167	0.04177	80	1.418	22	7.090	162	0.00409	7
1780	3188.4	396	57.76	171	0.04257	81	1.440	23	7.252	167	0.00416	7
1770	3228.0	399	59.47	174	0.04338	82	1.463	22	7.419	170	0.00423	7
1760	3267.9	401	61.21	179	0.04420	84	1.485	23	7.589	174	0.00430	8
1750	3308.0	403	63.00	183	0.04504	85	1.508	23	7.763	178	0.00438	7
1740	3348.3	406	64.83	188	0.04589	87	1.531	24	7.941	182	0.00445	8
1730	3388.9	409	66.71	193	0.04676	88	1.555	23	8.123	187	0.00453	8
1720	3429.8	410	68.64	197	0.04764	90	1.578	24	8.310	191	0.00461	8
1710	3470.8	413	70.61	202	0.04854	91	1.602	24	8.501	195	0.00469	9
1700	3512.1	415	72.63	207	0.04945	93	1.626	25	8.696	200	0.00478	8

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1690	3553.6	418	74.70	213	0.05038	95	1.651	25	8.896	205	0.00486	9
1680	3595.4	420	76.83	218	0.05133	96	1.676	25	9.101	210	0.00495	9
1670	3637.4	423	79.01	223	0.05229	98	1.701	25	9.311	215	0.00504	9
1660	3679.7	425	81.24	228	0.05327	100	1.726	26	9.526	220	0.00513	9
1650	3722.2	428	83.52	234	0.05427	102	1.752	26	9.746	225	0.00522	10
1640	3765.0	430	85.86	241	0.05529	103	1.778	26	9.971	231	0.00532	10
1630	3808.0	433	88.27	246	0.05632	106	1.804	27	10.202	237	0.00542	10
1620	3851.3	436	90.73	252	0.05738	107	1.831	27	10.439	243	0.00552	10
1610	3894.9	438	93.25	259	0.05845	110	1.858	27	10.682	249	0.00562	11
1600	3938.7	220	95.84	132	0.05955	55	1.885	14	10.931	127	0.00573	5
1595	3960.7	221	97.16	133	0.06010	56	1.899	14	11.058	128	0.00578	6
1590	3982.8	222	98.49	135	0.06066	57	1.913	14	11.186	130	0.00584	5
1585	4005.0	223	99.84	137	0.06123	57	1.927	14	11.316	132	0.00589	6
1580	4027.3	223	101.21	139	0.06180	58	1.941	14	11.448	133	0.00595	5
1575	4049.6	224	102.60	140	0.06238	58	1.955	14	11.581	135	0.00600	6
1570	4072.0	224	104.00	142	0.06296	59	1.969	14	11.716	137	0.00606	6
1565	4094.4	225	105.42	144	0.06355	59	1.983	15	11.853	138	0.00612	6
1560	4116.9	226	106.86	146	0.06414	60	1.998	14	11.991	140	0.00618	6
1555	4139.5	227	108.32	147	0.06474	60	2.012	15	12.131	143	0.00624	6
1550	4162.2	228	109.79	150	0.06534	61	2.027	15	12.274	144	0.00630	6
1545	4185.0	228	111.29	151	0.06595	62	2.042	15	12.418	145	0.00636	6
1540	4207.8	229	112.80	153	0.06657	62	2.057	15	12.563	148	0.00642	6
1535	4230.7	229	114.33	155	0.06719	63	2.072	14	12.711	150	0.00648	7
1530	4253.6	231	115.88	157	0.06782	64	2.086	15	12.861	152	0.00655	6
1525	4276.7	231	117.45	159	0.06846	64	2.101	16	13.013	153	0.00661	7
1520	4299.8	232	119.04	161	0.06910	65	2.117	15	13.166	156	0.00668	6
1515	4323.0	232	120.65	163	0.06975	65	2.132	15	13.322	158	0.00674	7
1510	4346.2	234	122.28	165	0.07040	66	2.147	15	13.480	160	0.00681	7
1505	4369.6	234	123.93	167	0.07106	67	2.162	16	13.640	162	0.00688	7
1500	4393.0	235	125.60	169	0.07173	68	2.178	16	13.802	164	0.00695	7
1495	4416.5	236	127.29	172	0.07241	68	2.194	16	13.966	166	0.00702	7
1490	4440.1	237	129.01	174	0.07309	69	2.210	16	14.132	169	0.00709	7
1485	4463.8	237	130.75	175	0.07378	69	2.226	16	14.301	171	0.00716	8

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1480	4487.5	238	132.50	178	0.07447	70	2.242	16	14.472	173	0.00724	7
1475	4511.3	239	134.28	181	0.07517	71	2.258	16	14.645	176	0.00731	8
1470	4535.2	240	136.09	183	0.07588	72	2.274	16	14.821	178	0.00739	7
1465	4559.2	240	137.92	185	0.07660	72	2.290	17	14.999	180	0.00746	8
1460	4583.2	242	139.77	188	0.07732	73	2.307	16	15.179	183	0.00754	8
1455	4607.4	242	141.65	189	0.07805	74	2.323	17	15.362	186	0.00762	8
1450	4631.6	243	143.54	193	0.07879	75	2.340	17	15.548	188	0.00770	8
1445	4655.9	244	145.47	195	0.07954	75	2.357	17	15.736	190	0.00778	8
1440	4680.3	245	147.42	197	0.08029	76	2.374	17	15.926	193	0.00786	8
1435	4704.8	246	149.39	200	0.08105	77	2.391	17	16.119	197	0.00794	8
1430	4729.4	247	151.39	203	0.08182	78	2.408	17	16.316	199	0.00802	8
1425	4754.1	247	153.42	205	0.08260	78	2.425	18	16.515	201	0.00810	9
1420	4778.8	248	155.47	208	0.08338	80	2.443	17	16.716	204	0.00819	9
1415	4803.6	249	157.55	211	0.08418	80	2.460	18	16.920	208	0.00828	9
1410	4828.5	250	159.66	214	0.08498	81	2.478	18	17.128	211	0.00837	9
1405	4853.5	251	161.80	216	0.08579	82	2.496	18	17.339	213	0.00846	9
1400	4878.6	252	163.96	219	0.08661	83	2.514	18	17.552	216	0.00855	9
1395	4003.8	253	166.15	222	0.08744	84	2.532	18	17.768	220	0.00864	10
1390	4929.1	254	168.37	225	0.08828	85	2.550	18	17.988	223	0.00874	9
1385	4954.5	254	170.62	228	0.08913	86	2.568	19	18.211	226	0.00883	10
1380	4979.9	256	172.90	231	0.08999	87	2.587	18	18.437	229	0.00893	9
1375	5005.5	256	175.21	234	0.09086	87	2.605	19	18.666	233	0.00902	10
1370	5031.1	257	177.55	237	0.09173	89	2.624	19	18.899	236	0.00912	10
1365	5056.8	258	179.92	241	0.09262	89	2.643	19	19.135	239	0.00922	11
1360	5082.6	260	182.33	243	0.09351	91	2.662	19	19.374	244	0.00933	10
1355	5108.6	260	184.76	247	0.09442	91	2.681	19	19.618	246	0.00943	11
1350	5134.6	261	187.23	250	0.09533	93	2.700	19	19.864	251	0.00954	10
1345	5160.7	262	189.73	254	0.09626	93	2.719	20	20.115	254	0.00964	11
1340	5186.9	263	192.27	257	0.09719	94	2.739	19	20.369	258	0.00975	11
1335	5213.2	263	194.84	260	0.09813	95	2.758	20	20.627	262	0.00986	11
1330	5239.5	263	197.44	262	0.09908	96	2.778	20	20.889	261	0.00997	11
1325	5265.8	262	200.06	263	0.10004	97	2.798	20	21.150	267	0.01008	12
1320	5292.0	106	202.69	107	0.10101	39	2.818	8	21.417	110	0.01020	5

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1318	5302.6	106	203.76	108	0.10140	39	2.826	8	21.527	109	0.01025	4
1316	5313.2	106	204.84	108	0.10179	40	2.834	8	21.636	110	0.01029	5
1314	5323.8	107	205.92	109	0.10219	40	2.842	8	21.746	109	0.01034	4
1312	5334.5	107	207.01	110	0.10259	40	2.850	8	21.855	110	0.01038	5
1310	5345.2	107	208.11	111	0.10299	40	2.858	8	21.965	112	0.01043	5
1308	5355.9	108	209.22	111	0.10339	41	2.866	9	22.077	113	0.01048	5
1306	5366.7	108	210.33	112	0.10380	41	2.875	8	22.190	114	0.01053	4
1304	5377.5	108	211.45	113	0.10421	41	2.883	9	22.304	115	0.01057	5
1302	5388.3	109	212.58	114	0.10462	41	2.892	8	22.419	116	0.01062	5
1300	5399.2	109	213.72	115	0.10503	41	2.900	8	22.535	116	0.01067	5
1298	5410.1	109	214.87	115	0.10544	42	2.908	9	22.651	118	0.01072	5
1296	5421.0	110	216.02	117	0.10586	42	2.917	8	22.769	118	0.01077	6
1294	5432.0	110	217.19	117	0.10628	42	2.925	9	22.887	120	0.01083	5
1292	5443.0	110	218.36	118	0.10670	43	2.934	8	23.007	120	0.01088	5
1290	5454.0	111	219.54	119	0.10713	43	2.942	8	23.127	121	0.01093	5
1288	5465.1	111	220.73	120	0.10756	43	2.950	9	23.248	122	0.01098	5
1286	5476.2	111	221.93	120	0.10799	43	2.959	9	23.370	124	0.01103	6
1284	5487.3	112	223.13	122	0.10842	44	2.968	9	23.494	124	0.01109	5
1282	5498.5	112	224.35	122	0.10886	44	2.977	8	23.618	125	0.01114	5
1280	5509.7	113	225.57	123	0.10930	44	2.985	9	23.743	126	0.01119	5
1278	5521.0	113	226.80	124	0.10974	45	2.994	9	23.869	127	0.01124	6
1276	5532.3	113	228.04	125	0.11019	45	3.003	9	23.996	129	0.01130	5
1274	5543.6	113	229.29	125	0.11064	45	3.012	9	24.125	129	0.01135	6
1272	5554.9	114	230.54	127	0.11109	45	3.021	9	24.254	130	0.01141	5
1270	5566.3	114	231.81	127	0.11154	46	3.030	9	24.384	131	0.01146	6
1268	5577.7	114	233.08	129	0.11200	46	3.039	9	24.515	132	0.01152	6
1266	5589.1	115	234.37	129	0.11246	46	3.048	9	24.647	134	0.01158	5
1264	5600.6	115	235.66	131	0.11292	46	3.057	9	24.781	134	0.01163	6
1262	5612.1	116	236.97	131	0.11338	47	3.066	9	24.915	136	0.01169	6
1260	5623.7	116	238.28	132	0.11385	47	3.075	9	25.051	136	0.01175	6
1258	5635.3	117	239.60	134	0.11432	47	3.084	10	25.187	138	0.01181	6
1256	5647.0	116	240.94	134	0.11479	48	3.094	9	25.325	139	0.01187	5
1254	5658.6	117	242.28	136	0.11527	48	3.103	10	25.464	140	0.01192	6

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1252	5670.3	118	243.64	136	0.11575	48	3.113	9	25.604	141	0.01198	6
1250	5682.1	118	245.00	137	0.11623	48	3.122	9	25.745	142	0.01204	6
1248	5693.9	118	246.37	139	0.11671	49	3.131	10	25.887	144	0.01210	6
1246	5705.7	119	247.76	139	0.11720	49	3.141	9	26.031	144	0.01216	7
1244	5717.6	119	249.15	140	0.11769	50	3.150	10	26.175	146	0.01223	6
1242	5729.5	119	250.55	142	0.11819	50	3.160	9	26.321	147	0.01229	6
1240	5741.4	120	251.97	142	0.11869	50	3.169	10	26.468	148	0.01235	6
1238	5753.4	120	253.39	144	0.11919	50	3.179	10	26.616	150	0.01241	7
1236	5765.4	121	254.83	144	0.11969	51	3.189	9	26.766	151	0.01248	6
1234	5777.5	121	256.27	146	0.12020	51	3.198	10	26.917	152	0.01254	7
1232	5789.6	121	257.73	147	0.12071	52	3.208	10	27.069	153	0.01261	6
1230	5801.7	122	259.20	148	0.12123	52	3.218	10	27.222	155	0.01267	7
1228	5813.9	122	260.68	149	0.12175	52	3.228	10	27.377	156	0.01274	6
1226	5826.1	123	262.17	150	0.12227	53	3.238	10	27.533	157	0.01280	7
1224	5838.4	123	263.67	151	0.12280	53	3.248	10	27.690	159	0.01287	6
1222	5850.7	123	265.18	153	0.12333	53	3.258	10	27.849	160	0.01293	7
1220	5863.0	124	266.71	153	0.12386	53	3.268	10	28.009	161	0.01300	7
1218	5875.4	124	268.24	155	0.12439	54	3.278	10	28.170	163	0.01307	7
1216	5887.8	125	269.79	156	0.12493	54	3.288	11	28.333	164	0.01314	7
1214	5900.3	125	271.35	157	0.12547	55	3.299	10	28.497	166	0.01321	7
1212	5912.8	125	272.92	159	0.12602	55	3.309	10	28.663	167	0.01328	7
1210	5925.3	126	274.51	160	0.12657	55	3.319	10	28.830	168	0.01335	7
1208	5937.9	126	276.11	161	0.12712	56	3.329	11	28.998	170	0.01342	7
1206	5950.5	127	277.72	162	0.12768	56	3.340	10	29.168	172	0.01349	8
1204	5963.2	127	279.34	163	0.12824	57	3.350	11	29.340	173	0.01357	7
1202	5975.9	127	280.97	165	0.12881	57	3.361	10	29.513	174	0.01364	7
1200	5988.6	128	282.62	166	0.12938	57	3.371	11	29.687	176	0.01371	8
1198	6001.4	128	284.28	167	0.12995	58	3.382	11	29.863	177	0.01379	7
1196	6014.2	129	285.95	168	0.13053	58	3.393	11	30.040	179	0.01386	8
1194	6027.1	129	287.63	170	0.13111	58	3.404	11	30.219	181	0.01394	7
1192	6040.0	130	289.33	171	0.13169	59	3.415	11	30.400	182	0.01401	8
1190	6053.0	130	291.04	172	0.13228	59	3.426	11	30.582	184	0.01409	8
1188	6066.0	131	292.76	174	0.13287	60	3.437	11	30.766	185	0.01417	8

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1186	6079.1	131	294.50	175	0.13347	60	3.448	11	30.951	187	0.01425	7
1184	6092.2	131	296.25	177	0.13407	60	3.459	11	31.138	189	0.01432	8
1182	6105.3	132	298.02	178	0.13467	61	3.470	11	31.327	190	0.01440	8
1180	6118.5	132	299.80	179	0.13528	61	3.481	11	31.517	192	0.01448	8
1178	6131.7	133	301.59	181	0.13589	62	3.492	12	31.709	194	0.01456	8
1176	6145.0	133	303.40	182	0.13651	62	3.504	11	31.903	195	0.01464	9
1174	6158.3	134	305.22	184	0.13713	63	3.515	12	32.098	198	0.01473	8
1172	6171.7	134	307.06	185	0.13776	63	3.527	11	32.296	199	0.01481	8
1170	6185.1	135	308.91	186	0.13839	63	3.538	12	32.495	201	0.01489	9
1168	6198.6	135	310.77	188	0.13902	64	3.550	11	32.696	203	0.01498	8
1166	6212.1	135	312.65	190	0.13966	64	3.561	12	32.899	205	0.01506	9
1164	6225.6	136	314.55	191	0.14030	65	3.573	11	33.104	206	0.01515	8
1162	6239.2	136	316.46	193	0.14095	65	3.584	12	33.310	209	0.01523	9
1160	6252.8	69	318.39	97	0.14160	32	3.596	6	33.519	105	0.01532	4
1159	6259.7	69	319.36	98	0.14192	33	3.602	6	33.624	106	0 01536	5
1158	6266.6	68	320.34	98	0.14225	33	3.608	6	33 730	106	0.01541	4
1157	6273.4	69	321.32	98	0.14258	33	3.614	6	33.836	106	0.01545	5
1156	6280.3	69	322.30	98	0.14291	33	3.620	6	33.942	107	0.01550	4
1155	6287.2	69	323.28	99	0.14324	34	3.626	6	34.049	107	0.01554	5
1154	6294.1	69	324.27	99	0.14358	33	3.632	6	34.156	108	0.01559	4
1153	6301.0	69	325.26	100	0.14391	34	3.638	6	34.264	108	0.01563	5
1152	6307.9	69	326.26	100	0.14425	33	3.644	6	34.372	109	0.01568	4
1151	6314.8	70	327.26	101	0.14458	34	3.650	6	34.481	109	0.01572	5
1150	6321.8	70	328.27	101	0.14492	34	3.656	6	34.590	110	0.01577	5
1149	6328.8	69	329.28	101	0.14526	34	3.662	6	34.700	111	0.01582	4
1148	6335.7	70	330.29	102	0.14560	34	3.668	6	34.811	111	0.01586	5
1147	6342.7	70	331.31	102	0.14594	34	3.674	6	34.922	112	0.01591	5
1146	6349.7	70	332.33	103	0.14628	34	3.680	6	35.034	112	0.01596	5
1145	6356.7	70	333.36	103	0.14662	35	3.686	7	35.146	112	0.01601	4
1144	6363.7	70	334.39	104	0.14697	34	3.693	6	35.258	113	0.01605	5
1143	6370.7	71	335.43	104	0.14731	35	3.699	6	35.371	113	0.01610	5
1142	6377.8	70	336.47	104	0.14766	35	3.705	6	35.484	114	0.01615	4
1141	6384.8	71	337.51	105	0.14801	35	3.711	6	35.598	114	0 01619	5

TABLE 1.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1140	6391.9	71	338.56	105	0.14836	35	3.717	6	35.712	115	0.01624	5
1139	6399.0	71	339.61	106	0.14871	35	3.723	7	35.827	116	0.01629	5
1138	6406.1	71	340.67	106	0.14906	36	3.730	6	35.943	116	0.01634	4
1137	6413.2	71	341.73	106	0.14942	35	3.736	6	36.059	117	0.01638	5
1136	6420.3	71	342.79	107	0.14977	36	3 742	6	36 176	117	0.01643	5
1135	6427.4	72	343.86	108	0.15013	36	3.748	7	36.293	118	0.01648	5
1134	6434.6	71	344.94	108	0.15049	36	3.755	6	30.411	119	0.01653	5
1133	6441.7	72	346.02	108	0.15085	36	3.761	6	36.530	119	0.01658	4
1132	6448.9	72	347.10	109	0.15121	36	3.767	7	36.649	120	0.01662	5
1131	6456.1	72	348.19	109	0.15157	36	3.774	6	36.769	120	0.01667	5
1130	6463.3	71	349.28	110	0.15193	36	3.780	6	36.889	121	0.01672	5
1129	6470.4	72	350.38	109	0.15229	36	3.786	7	37.010	121	0.01677	5
1128	6477.6	72	351.47	110	0.15265	37	3.793	6	37.131	122	0.01682	5
1127	6484.8	73	352.57	111	0.15302	36	3.799	7	37.253	122	0.01687	5
1126	6492.1	72	353.68	111	0.15338	37	3.806	6	37.375	123	0.01692	5
1125	6499.3	73	354.79	111	0.15375	37	3.812	6	37.498	124	0.01697	6
1124	6506.6	73	355.90	113	0.15412	37	3.818	7	37.622	124	0.01703	5
1123	6513.9	73	357.03	113	0.15449	38	3.825	6	37.746	125	0.01708	5
1122	6521.2	74	358.16	114	0.15487	37	3.831	7	37.871	125	0.01713	5
1121	6528.6	74	359.30	115	0.15524	38	3.838	6	37.996	126	0.01718	5
1120	6536.0	74	360.45	115	0.15562	38	3.844	7	38.122	128	0.01723	5
1119	6543.4	74	361.60	116	0.15600	38	3.851	7	38.250	128	0.01728	6
1118	6550.8	75	362.76	116	0.15638	38	3.858	6	38.378	130	0.01734	5
1117	6558.3	75	363.92	117	0.15676	39	3.864	7	38.508	130	0.01739	6
1116	6565.8	75	365.09	119	0.15715	39	3.871	7	38.638	132	0.01745	5
1115	6573.3	75	366.28	119	0.15754	39	3.878	7	38.770	132	0.01750	6
1114	6580.8	76	367.47	120	0.15793	39	3.885	7	38.902	134	0 01756	5
1113	6588.4	76	368.67	121	0.15832	40	3.892	6	39.036	134	0.01761	6
1112	6596.0	77	369.88	121	0.15872	40	3.898	7	39.170	136	0.01767	5
1111	6603.7	77	371.09	123	0.15912	40	3.905	7	39.306	136	0.01772	6
1110	6611.4	77	372.32	123	0.15952	41	3.912	7	39.442	138	0.01778	6
1109	6619.1	78	373.55	124	0.15993	40	3.919	7	39.580	138	0.01784	6
1108	6626.9	78	374.79	125	0.16033	41	3.926	7	39.718	140	0.01790	5

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1107	6634.7	78	376.04	126	0.16074	41	3.933	7	39.858	140	0.01795	6
1106	6642.5	78	377.30	127	0.16115	42	3.940	7	39.998	142	0.01801	6
1105	6650.3	79	378.57	128	0.16157	41	3.947	8	40.140	143	0.01807	6
1104	6658.2	80	379.85	129	0.16198	42	3.955	7	40.283	144	0.01813	5
1103	6666.2	79	381.14	130	0.16240	42	3.962	7	40.427	145	0.01818	6
1102	6674.1	80	382.44	131	0.16282	43	3.969	7	40.572	147	0.01824	6
1101	6682.1	81	383.75	131	0.16325	42	3.976	7	40.719	147	0.01830	6
1100	6690.2	81	385.06	132	0.16367	43	3.983	8	40.866	149	0.01836	6
1099	6698.3	81	386.38	133	0.16410	43	3.991	7	41.015	150	0.01842	7
1098	6706.4	81	387.71	135	0.16453	44	3.998	8	41.165	151	0.01849	6
1097	6714.5	82	389.06	135	0.16497	44	4.006	7	41.316	152	0.01855	6
1096	6722.7	83	390.41	137	0.16541	44	4.013	8	41.468	153	0.01861	6
1095	6731.0	82	391.78	137	0.16585	44	4.021	8	41.621	155	0.01867	7
1094	6739.2	83	393.15	138	0.16629	45	4.029	7	41.776	156	0.01874	6
1093	6747.5	84	394.53	140	0.16674	45	4.036	8	41.932	157	0.01880	6
1092	6755.9	84	395.93	141	0.16719	45	4.044	7	42.089	158	0.01886	7
1091	6764.3	84	397.34	141	0.16764	46	4.051	8	42.247	160	0.01893	6
1090	6772.7	85	398.75	142	0.16810	46	4.059	8	42.407	161	0.01899	7
1089	6781.2	85	400.17	143	0.16856	46	4.067	8	42.568	162	0.01906	7
1088	6789.7	85	401.60	145	0.16902	46	4.075	8	42.730	164	0.01913	6
1087	6798.2	86	403.05	145	0.16948	47	4.083	8	42.894	165	0.01919	7
1086	6806.8	86	404.50	147	0.16995	47	4.091	7	43.059	166	0.01926	7
1085	6815.4	87	405.97	148	0.17042	47	4.098	8	43.225	168	0.01933	7
1084	6824.1	87	407.45	149	0.17089	48	4.106	8	43.393	169	0.01940	7
1083	6832.8	87	408.94	150	0.17137	48	4.114	8	43.562	170	0.01947	6
1082	6841.5	88	410.44	151	0.17185	48	4.122	8	43.732	172	0.01953	7
1081	6850.3	88	411.95	152	0.17233	49	4.130	8	43.904	173	0.01960	7
1080	6859.1	88	413.47	153	0.17282	49	4.138	8	44.077	175	0.01967	7
1079	6867.9	89	415.00	154	0.17331	49	4.146	9	44.252	176	0.01974	8
1078	6876.8	90	416.54	156	0.17380	49	4.155	8	44.428	177	0.01982	7
1077	6885.8	89	418.10	156	0.17429	50	4.163	9	44.605	179	0.01989	7
1076	6894.7	90	419.66	158	0.17479	50	4.172	8	44.784	180	0.01996	7
1075	6903.7	91	421.24	159	0.17529	51	4.180	9	44.964	182	0.02003	8

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1074	6912.8	91	422.83	161	0.17580	51	4.189	8	45.146	184	0.02011	7
1073	6921.9	92	424.44	162	0.17631	51	4.197	9	45.330	185	0.02018	7
1072	6931.1	92	426.06	163	0.17682	51	4.206	8	45.515	186	0.02025	8
1071	6940.3	92	427.69	164	0.17733	52	4.214	9	45.701	188	0.02033	7
1070	6949.5	93	429.33	165	0.17785	52	4.223	9	45.889	190	0.02040	8
1069	6958.8	93	430.98	166	0.17837	53	4.232	9	46.079	191	0.02048	8
1068	6968.1	94	432.64	168	0.17890	53	4.241	9	46.270	193	0.02056	7
1067	6977.5	94	434.32	169	0.17943	53	4.250	9	46.463	194	0.02063	8
1066	6986.9	94	436.01	171	0.17996	53	4.259	9	46.657	196	0.02071	8
1065	6996.3	95	437.72	172	0.18049	54	4.268	9	46.853	198	0.02079	8
1064	7005.8	96	439.44	173	0.18103	55	4.277	9	47.051	199	0.02087	8
1063	7015.4	96	441.17	175	0.18158	55	4.286	9	47.250	201	0.02095	7
1062	7025.0	96	442.92	176	0.18213	55	4.295	9	47.451	203	0.02102	8
1061	7034.6	97	444.68	177	0.18268	55	4.304	9	47.654	205	0.02110	8
1060	7044.3	97	446.45	178	0.18323	56	4.313	9	47.859	207	0.02118	8
1059	7054.0	98	448.23	180	0.18379	56	4.322	10	48.066	208	0.02126	9
1058	7063.8	98	450.03	181	0.18435	56	4.332	9	48.274	210	0.02135	8
1057	7073.6	99	451.84	182	0.18491	57	4.341	9	48.484	212	0.02143	9
1056	7083.5	99	453.66	184	0.18548	57	4.350	10	48.696	213	0.02152	8
1055	7093.4	100	455.50	186	0.18605	58	4.360	9	48.909	216	0.02160	9
1054	7103.4	100	457.36	187	0.18663	58	4.369	9	49.125	217	0.02169	8
1053	7113.4	100	459.23	189	0.18721	58	4.378	9	49.342	219	0.02177	9
1052	7123.4	101	461.12	190	0.18779	59	4.387	10	49.561	222	0.02186	8
1051	7133.5	102	463.02	192	0.18838	59	4.397	9	49.783	223	0.02194	9
1050	7143.7	102	464.94	193	0.18897	59	4.406	10	50.006	225	0.02203	9
1049	7153.9	102	466.87	194	0.18956	60	4.416	10	50.231	228	0.02212	9
1048	7164.1	103	468.81	196	0.19016	61	4.426	10	50.459	229	0.02221	9
1047	7174.4	103	470.77	197	0.19077	61	4.436	10	50.688	231	0.02230	9
1046	7184.7	104	472.74	199	0.19138	61	4.446	9	50.919	233	0.02239	9
1045	7195.1	105	474.73	201	0.19199	61	4.455	10	51.152	235	0.02248	10
1044	7205.6	105	476.74	203	0.19260	62	4.465	10	51.387	237	0.02258	9
1043	7216.1	105	478.77	204	0.19322	63	4.475	10	51.624	240	0.02267	9
1042	7226.6	106	480.81	206	0.19385	63	4.485	10	51.864	241	0.02276	9

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1041	7237.2	107	482.87	208	0.19448	63	4.495	10	52.105	244	0.02285	9
1040	7247.9	107	484.95	209	0.19511	64	4.505	11	52.349	246	0.02294	10
1039	7258.6	107	487.04	211	0.19575	64	4.516	10	52.595	248	0.02304	10
1038	7269.3	108	489.15	213	0.19639	64	4.526	11	52.843	250	0.02314	10
1037	7280.1	109	491.28	214	0.19703	65	4.537	10	53.093	253	0.02324	10
1036	7291.0	109	493.42	216	0.19768	66	4.547	11	53.346	255	0.02334	10
1035	7301.9	110	495.58	218	0.19834	66	4.558	11	53.601	257	0.02344	9
1034	7312.9	110	497.76	219	0.19900	66	4.569	10	53.858	260	0.02353	10
1033	7323.9	111	499.95	222	0.19966	67	4.579	11	54.118	262	0.02363	10
1032	7335.0	111	502.17	223	0.20033	67	4.590	10	54.380	264	0.02373	10
1031	7346.1	112	504.40	225	0.20100	68	4.600	11	54.644	267	0.02383	10
1030	7357.3	112	506.65	226	0.20168	68	4.611	11	54.911	269	0.02393	11
1029	7368.5	113	508.91	229	0.20236	69	4.622	11	55.180	272	0.02404	10
1028	7379.8	113	511.20	230	0.20305	69	4.633	12	55.452	274	0.02414	11
1027	7391.1	114	513.50	232	0.20374	69	4.645	11	55.726	277	0.02425	10
1026	7402.5	115	515.82	235	0.20443	70	4.656	11	56.003	280	0.02435	11
1025	7414.0	115	518.17	237	0.20513	71	4.667	11	56.283	282	0.02446	11
1024	7425.5	116	520.54	238	0.20584	71	4.678	11	56.565	284	0.02457	10
1023	7437.1	116	522.92	240	0.20655	71	4.689	12	56.849	287	0.02467	11
1022	7448.7	117	525.32	243	0.20726	72	4.701	11	57.136	290	0.02478	10
1021	7460.4	117	527.75	245	0.20798	73	4.712	11	57.426	293	0.02488	11
1020	7472.1	118	530.20	246	0.20871	73	4.723	12	57.719	296	0.02499	11
1019	7483.9	118	532.66	248	0.20944	73	4.735	12	58.015	298	0.02510	12
1018	7495.7	119	535.14	251	0.21017	74	4.747	12	58.313	301	0.02522	12
1017	7507.6	120	537.65	252	0.21091	74	4.759	12	58.614	304	0.02534	11
1016	7519.6	120	540.17	255	0.21165	75	4.771	11	58.918	306	0.02545	12
1015	7531.6	121	542.72	258	0.21240	76	4.782	12	59.224	310	0.02557	12
1014	7543.7	121	545.30	259	0.21316	76	4.794	12	59.534	312	0.02569	11
1013	7555.8	122	547.89	262	0.21392	76	4.806	12	59.846	316	0.02580	12
1012	7568.0	123	550.51	265	0.21468	77	4.818	12	60.162	318	0.02592	11
1011	7580.3	123	553.16	266	0.21545	78	4.830	12	60.480	322	0.02603	12
1010	7592.6	124	555.82	269	0.21623	78	4.842	13	60.802	325	0.02615	12
1009	7605.0	124	558.51	272	0.21701	79	4.855	12	61.127	328	0.02627	13

TABLE 1.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
1008	7617.4	125	561.23	273	0.21780	79	4.867	13	61.455	331	0.02640	12
1007	7629.9	126	563.96	275	0.21859	80	4.880	12	61.786	334	0.02652	13
1006	7642.5	126	566.71	278	0.21939	80	4.892	13	62.120	337	0.02665	12
1005	7655.1	127	569.49	280	0.22019	81	4.905	13	62.457	340	0.02677	12
1004	7667.8	128	572.29	282	0.22100	82	4.918	12	62.797	344	0.02689	13
1003	7680.6	128	575.11	285	0.22182	82	4.930	13	63.141	347	0.02702	12
1002	7693.4	129	577.96	287	0.22264	83	4.943	12	63.488	351	0.02714	13
1001	7706.3	130	580.83	289	0.22347	83	4.955	13	63.839	354	0.02727	12
1000	7719.3	131	583.72	292	0.22430	84	4.968	13	64.193	358	0.02739	13
999	7732.4	132	586.64	295	0.22514	85	4.981	14	64.551	361	0.02752	14
998	7745.6	132	589.59	297	0.22599	85	4.995	13	64.912	364	0.02766	14
997	7758.8	133	592.56	300	0.22684	86	5.008	14	65.276	368	0.02780	13
996	7772.1	133	595.56	303	0.22770	87	5.022	13	65.644	372	0.02793	14
995	7785.4	133	598.59	306	0.22857	87	5.035	13	66.016	376	0.02807	14
994	7798.7	134	601.65	309	0.22944	87	5.048	14	66.392	379	0.02821	13
993	7812.1	134	604.74	311	0.23031	87	5.062	13	66.771	383	0.02834	14
992	7825.5	135	607.85	314	0.23118	88	5.075	14	67.154	386	0.02848	13
991	7839.0	135	610.99	317	0.23206	89	5.089	13	67.540	391	0.02861	14
990	7852.5	136	614.16	317	0.23295	89	5.102	14	67.931	392	0.02875	14
989	7866.1	136	617.33	319	0.23384	90	5.116	14	68.323	394	0.02889	15
988	7879.7	137	620.52	321	0.23474	90	5.130	14	68.717	398	0.02904	14
987	7893.4	137	623.73	323	0.23564	91	5.144	14	69.115	400	0.02918	14
986	7907.1	137	626.96	325	0.23655	91	5.158	13	69.515	403	0.02932	14
985	7920.8	137	630.21	327	0.23746	91	5.171	14	69.918	406	0.02946	15
984	7934.5	138	633.48	329	0.23837	92	5.185	14	70.324	408	0.02961	14
983	7948.3	138	636.77	331	0.23929	92	5.199	14	70.732	412	0.02975	14
982	7962.1	138	640.08	333	0.24021	92	5.213	14	71.144	414	0.02989	15
981	7975.9	139	643.41	335	0.24113	93	5.227	14	71.558	417	0.03004	14
980	7989.8	139	646.76	336	0.24206	93	5.241	14	71.975	420	0.03018	15
979	8003.7	139	650.12	339	0.24299	93	5.255	15	72.395	423	0.03033	15
978	8017.6	139	653.51	341	0.24392	94	5.270	14	72.818	427	0.03048	16
977	8031.5	140	656.92	343	0.24486	94	5.284	15	73.245	429	0.03064	15
976	8045.5	140	660.35	345	0.24580	95	5.299	14	73.674	432	0.03079	15

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
975	8059.5	140	663.80	346	0.24675	95	5.313	14	74.106	435	0.03094	15
974	8073.5	141	667.26	349	0.24770	95	5.327	15	74.541	438	0.03109	15
973	8087.6	141	670.75	351	0.24865	96	5.342	14	74.979	441	0.03124	16
972	8101.7	141	674.26	354	0.24961	96	5.356	15	75.420	444	0.03140	15
971	8115.8	141	677.80	355	0.25057	97	5.371	14	75.864	447	0.03155	15
970	8129.9	142	681.35	357	0.25154	97	5.385	15	76.311	450	0.03170	16
969	8144.1	142	684.92	359	0.25251	97	5.400	15	76.761	453	0.03186	16
968	8158.3	142	688.51	361	0.25348	98	5.415	14	77.214	457	0.03202	16
967	8172.5	143	692.12	363	0.25446	98	5.429	15	77.671	460	0.03218	16
966	8186.8	143	695.75	366	0.25544	99	5.444	15	78.131	463	0.03234	16
965	8201.1	143	699.41	368	0.25643	99	5.459	15	78.594	466	0.03250	15
964	8215.4	144	703.09	370	0.25742	99	5.474	15	79.060	470	0.03265	16
963	8229.8	144	706.79	372	0.25841	100	5.489	14	79.530	473	0.03281	16
962	8244.2	144	710.51	375	0.25941	100	5.503	15	80.003	476	0.03297	16
961	8258.6	144	714.26	377	0.26041	101	5.518	15	80.479	479	0.03313	16
960	8273.0	144	718.03	378	0.26142	101	5.533	15	80.958	483	0.03329	17
959	8287.4	145	721.81	381	0.26243	101	5.548	16	81.441	486	0.03346	16
958	8301.9	145	725.62	384	0.26344	102	5.564	15	81.927	490	0.03362	17
957	8316.4	146	729.46	386	0.26446	103	5.579	15	82.417	493	0.03379	17
956	8331.0	146	733.32	388	0.26549	103	5.594	15	82.910	497	0.03396	16
955	8345.6	146	737.20	390	0.26652	103	5.609	16	83.407	500	0.03412	17
954	8360.2	146	741.10	393	0.26755	103	5.625	15	83.907	503	0.03429	17
953	8374.8	147	745.03	395	0.26858	104	5.640	15	84.410	507	0.03446	17
952	8389.5	147	748.98	398	0.26962	105	5.655	16	84.917	511	0.03463	16
951	8404.2	148	752.96	400	0.27067	105	5.671	15	85.428	514	0.03479	17
950	8419.0	148	756.96	402	0.27172	105	5.686	16	85.942	518	0.03496	18
949	8433.8	148	760.98	404	0.27277	106	5.702	16	86.460	521	0.03514	17
948	8448.6	148	765.02	407	0.27383	106	5.718	15	86.981	525	0.03531	18
947	8463.4	149	769.09	409	0.27489	107	5.733	16	87.506	529	0.03549	18
946	8478.3	149	773.18	412	0.27596	107	5.749	16	88.035	533	0.03567	17
945	8493.2	149	777.30	415	0.27703	108	5.765	16	88.568	537	0.03584	18
944	8508.1	150	781.45	417	0.27811	108	5.781	16	89.105	540	0.03602	18
943	8523.1	150	785.62	420	0.27919	108	5.797	15	89.645	544	0.03620	18

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
942	8538.1	150	789.82	422	0.28027	109	5.812	16	90.189	547	0.03638	17
941	8553.1	151	794.04	425	0.28136	110	5.828	16	90.736	552	0.03655	18
940	8568.2	151	798.29	427	0.28246	110	5.844	16	91.288	556	0.03673	19
939	8583.3	151	802.56	429	0.28356	111	5.860	17	91.844	559	0.03692	18
938	8598.4	152	806.85	432	0.28467	111	5.877	16	92.403	564	0.03710	19
937	8613.6	152	811.17	435	0.28578	111	5.893	16	92.967	568	0.03729	19
936	8628.8	152	815.52	437	0.28689	112	5.909	17	93.535	572	0.03748	18
935	8644.0	152	819.89	441	0.28801	112	5.926	16	94.107	576	0.03766	19
934	8659.2	153	824.30	443	0.28913	113	5.942	16	94.683	580	0.03785	19
933	8674.5	153	828.73	445	0.29026	114	5.958	16	95.263	584	0.03804	19
932	8689.8	154	833.18	449	0.29140	114	5.974	17	95.847	588	0.03823	18
931	8705.2	154	837.67	451	0.29254	114	5.991	16	96.435	592	0.03841	19
930	8720.6	154	842.18	453	0.29368	115	6.007	17	97.027	596	0.03860	20
929	8736.0	155	846.71	456	0.29483	115	6.024	17	97.623	601	0.03880	19
928	8751.5	155	851.27	459	0.29598	116	6.041	16	98.224	605	0.03899	20
927	8767.0	155	855.86	462	0.29714	116	6.057	17	98.829	610	0.03919	19
926	8782.5	155	860.48	465	0.29830	117	6.074	17	99.439	615	0.03938	20
925	8798.0	156	865.13	468	0.29947	117	6.091	17	100.05	62	0.0396	2
924	8813.6	156	869.81	470	0.30064	118	6.108	17	100.67	62	0.0398	2
923	8829.2	157	874.51	474	0.30182	118	6.125	16	101.29	63	0.0400	2
922	8844.9	157	879.25	477	0.30300	119	6.141	17	101.92	63	0.0402	2
921	8860.6	157	884.02	479	0.30419	119	6.158	17	102.55	64	0.0404	2
920	8876.3	157	888.81	482	0.30538	120	6.175	17	103.19	64	0.0406	2
919	8892.0	158	893.63	485	0.30658	120	6.192	18	103.83	65	0.0408	2
918	8907.8	159	898.48	488	0.30778	121	6.210	17	104.48	65	0.0410	2
917	8923.7	158	903.36	491	0.30899	121	6.227	18	105.13	66	0.0412	2
916	8939.5	159	908.27	494	0.31020	122	6.245	17	105.79	66	0.0414	2
915	8955.4	159	913.21	497	0.31142	122	6.262	17	106.45	66	0.0416	2
914	8971.3	160	918.18	501	0.31264	123	6.279	18	107.11	67	0.0418	2
913	8987.3	160	923.19	503	0.31387	124	6.297	17	107.78	67	0.0420	2
912	9003.3	160	928.22	506	0.31511	124	6.314	18	108.45	68	0.0422	2
911	9019.3	161	933.28	509	0.31635	125	6.332	17	109.13	68	0.0424	2
910	9035.4	161	948.37	513	0.31760	125	6.349	18	109.81	68	0.0426	2

TABLE 1.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
909	9051.5	161	943.50	515	0.31885	126	6.367	18	110.49	69	0.0428	2
908	9067.6	162	948.65	519	0.32011	126	6.385	18	111.18	70	0.0430	3
907	9083.8	162	953.84	522	0.32137	127	6.403	18	111.88	70	0.0433	2
906	9100.0	162	959.06	525	0.32264	128	6.421	18	112.58	71	0.0435	2
905	9116.2	163	964.31	529	0.32392	128	6.439	18	113.29	71	0.0437	2
904	9132.5	163	969.60	532	0.32520	129	6.457	18	114.00	72	0.0439	2
903	9148.8	164	974.92	535	0.32649	129	6.475	18	114.72	73	0.0441	3
902	9165.2	164	980.27	538	0.32778	130	6.493	18	115.45	73	0.0444	2
901	9181.6	164	985.65	541	0.32908	130	6.511	18	116.18	74	0.0446	2
900	9198.0	165	991.06	545	0.33038	131	6.529	19	116.92	74	0.0448	3
899	9214.5	165	996.51	548	0.33169	131	6.548	18	117.66	74	0.0451	2
898	9231.0	165	1001.99	552	0.33300	132	6.566	19	118 40	75	0.0453	3
897	9247.5	166	1007.51	555	0.33432	133	6.585	18	119.15	76	0.0456	2
896	9264.1	166	1013.06	559	0.33565	133	6.603	19	119.91	76	0.0458	2
895	9280.7	166	1018.65	562	0.33698	134	6.622	18	120.67	77	0.0460	3
894	9297.3	167	1024.27	565	0.33832	134	6.640	19	121.44	77	0.0463	2
893	9314.0	167	1029.92	569	0.33966	135	6.659	18	122.21	78	0.0465	3
892	9330.7	168	1035.61	573	0.34101	136	6.677	19	122.99	79	0.0468	2
891	9347.5	168	1041.34	576	0.34237	136	6.696	18	123.78	79	0.0470	2
890	9364.3	168	1047.10	580	0.34373	137	6.714	19	124.57	79	0.0472	2
889	9381.1	169	1052.90	583	0.34510	137	6.733	20	125.36	80	0.0474	3
888	9398.0	169	1058.73	587	0.34647	138	6.753	19	126.16	81	0.0477	2
887	9414.9	170	1064.60	592	0.34785	139	6.772	19	126.97	81	0.0479	3
886	9431.9	170	1070.52	595	0.34924	139	6.791	20	127.78	82	0 0482	2
885	9448.9	170	1076.47	598	0.35063	140	6.811	19	128.60	83	0.0484	2
884	9465.9	171	1082.45	602	0.35203	141	6.830	19	129.43	83	0.0486	3
883	9483.0	171	1088.47	606	0.35344	141	6.849	19	130.26	84	0.0489	2
882	9500.1	171	1094.53	609	0.35485	142	6.868	20	131.10	84	0.0491	3
881	9517.2	172	1100.62	613	0.35627	143	6.888	19	131.94	85	0.0494	2
880	9534.4	172	1106.75	617	0.35770	143	6.907	20	132.79	86	0.0496	3
879	9551.6	173	1112.92	621	0.35913	144	6.927	20	133.65	86	0.0499	2
878	9568.9	173	1119.13	625	0.36057	145	6.947	19	134.51	87	0.0501	3
877	9586.2	173	1125.38	629	0.36202	145	6.966	20	135.38	88	0.0504	2

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
876	9603.5	174	1131.67	633	0.36347	146	6.986	20	136.26	88	0.0506	3
875	9620.9	174	1138.00	637	0.36493	146	7.006	20	137.14	89	0.0509	3
874	9638.3	175	1144.37	641	0.36639	147	7.026	20	138.03	90	0.0512	2
873	9655.8	175	1150.78	645	0.36786	148	7.046	19	138.93	90	0.0514	3
872	9673.3	175	1157.23	649	0.36934	149	7.065	20	139.83	91	0.0517	2
871	9690.8	176	1163.72	653	0.37083	149	7.085	20	140.74	91	0.0519	3
870	9708.4	176	1170.25	657	0.37232	150	7.105	21	141.65	92	0.0522	3
869	9726.0	177	1176.82	662	0.37382	150	7.126	20	142.57	92	0.0525	3
868	9743.7	177	1183.44	665	0.37532	151	7.146	21	143.49	94	0.0528	2
867	9761.4	177	1190.09	670	0.37683	152	7.167	20	144.43	94	0.0530	3
866	9779.1	178	1196.79	675	0.37835	153	7.187	21	145.37	95	0.0533	3
865	9796.9	178	1203.54	678	0.37988	153	7.208	21	146.32	96	0.0536	3
864	9814.7	179	1210.32	683	0.38141	154	7.229	20	147.28	96	0.0539	3
863	9832.6	179	1217.15	687	0.38295	155	7.249	21	148.24	98	0.0542	2
862	9850.5	179	1224.02	691	0.38450	156	7.270	20	149.22	98	0.0544	3
861	9868.4	180	1230.93	696	0.38606	156	7.290	21	150.20	99	0.0547	3
860	9886.4	180	1237.89	700	0.38762	157	7.311	21	151.19	99	0.0550	3
859	9904.4	181	1244.89	705	0.38919	158	7.332	22	152.18	100	0.0553	3
858	9922.5	181	1251.94	710	0.39077	158	7.354	21	153.18	101	0.0556	3
857	9940.6	181	1259.04	714	0.39235	159	7.375	21	154.19	102	0.0559	3
856	9958.7	182	1266.18	718	0.39394	160	7.396	22	155.21	102	0.0562	2
855	9976.9	183	1273.36	723	0.39554	161	7.418	21	156.23	103	0.0564	3
854	9995.2	183	1280.59	728	0.39715	162	7.439	21	157.26	104	0.0567	3
853	10013.5	183	1287.87	732	0.39877	162	7.460	21	158.30	105	0.0570	3
852	10031.8	184	1295.19	737	0.40039	163	7.481	22	159.35	106	0.0573	3
851	10050.2	184	1302.56	742	0.40202	164	7.503	21	160.41	106	0.0576	3
850	10068.6	185	1309.98	746	0.40366	164	7.524	22	161.47	107	0.0579	3
849	10087.1	185	1317.44	752	0.40530	165	7.546	22	162.54	108	0.0582	3
848	10105.6	185	1324.96	756	0.40695	166	7.568	22	163.62	109	0.0585	3
847	10124.1	186	1332.52	761	0.40861	167	7.590	22	164.71	110	0.0588	3
846	10142.7	186	1340.13	766	0.41028	168	7.612	23	165.81	110	0.0591	3
845	10161.3	187	1347.79	771	0.41196	168	7.635	22	166.91	111	0.0594	4
844	10180.0	188	1355.50	776	0.41364	169	7.657	22	168.02	112	0.0598	3

TABLE 1.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
843	10198.8	187	1363.26	781	.41533	170	7.679	22	169.14	113	.0601	3
842	10217.5	188	1371.07	786	.41703	171	7.701	22	170.27	114	.0604	3
841	10236.3	189	1378.93	791	.41874	172	7.723	22	171.41	114	.0607	3
840	10255.2	189	1386.84	796	.42046	172	7.745	23	172.55	115	.0610	3
839	10274.1	189	1394.80	802	.42218	174	7.768	22	173.70	116	.0613	3
838	10293.0	190	1402.82	807	.42392	174	7.790	23	174.86	117	.0616	4
837	10312.0	190	1410.89	812	.42566	175	7.813	23	176.03	118	.0620	3
836	10331.0	191	1419.01	817	.42741	176	7.836	22	177.21	120	.0623	3
835	10350.1	191	1427.18	823	.42917	176	7.858	23	178.41	120	.0626	3
834	10369.2	192	1435.41	828	.43093	178	7.881	23	179.61	121	.0629	3
833	10388.4	192	1443.69	833	.43271	178	7.904	24	180.82	122	.0632	4
832	10407.6	193	1452.02	839	.43449	180	7.928	23	182.04	123	.0636	3
831	10426.9	193	1460.41	844	.43629	180	7.951	23	183.27	124	.0639	3
830	10446.2	194	1468.85	850	.43809	181	7.974	23	184.51	125	.0642	3
829	10465.6	194	1477.35	855	.43990	182	7.997	24	185.76	126	.0645	4
828	10485.0	194	1485.90	861	.44172	182	8.021	23	187.02	126	.0649	3
827	10504.4	195	1494.51	867	.44354	184	8.044	24	188.28	127	.0652	4
826	10523.9	195	1503.18	872	.44538	184	8.068	23	189.55	129	.0656	3
825	10543.4	196	1511.90	879	.44722	186	8.091	24	190.84	130	.0659	4
824	10563.0	197	1520.69	883	.44908	186	8.115	24	192.14	130	.0663	3
823	10582.7	197	1529.52	890	.45094	188	8.139	24	193.44	132	.0666	4
822	10602.4	197	1538.42	896	.45282	188	8.163	24	194.76	132	.0670	3
821	10622.1	198	1547.38	901	.45470	189	8.187	24	196.08	134	.0673	4
820	10641.9	198	1556.39	908	.45659	190	8.211	24	197.42	135	.0677	4
819	10661.7	199	1565.47	914	.45849	191	8.235	24	198.77	135	.0681	3
818	10681.6	200	1574.61	919	.46040	191	8.259	25	200.12	136	.0684	4
817	10701.6	200	1583.80	925	.46231	193	8.284	24	201.48	138	.0688	4
816	10721.6	200	1593.05	932	.46424	194	8.308	25	202.86	139	.0692	3
815	10741.6	201	1602.37	938	.46618	194	8.333	24	204.25	140	.0695	4
814	10761.7	201	1611.75	945	.46812	196	8.357	25	205.65	142	.0699	4
813	10781.8	202	1621.20	950	.47008	197	8.382	25	207.07	142	.0703	4
812	10802.0	202	1630.70	957	.47205	197	8.407	25	208.49	144	.0707	3
811	10822.2	203	1640.27	963	.47402	199	8.432	25	209.93	145	.0710	4

TABLE I.—Continued.

u	S(u)	Diff	A(u)	Diff	I(u)	Diff	T(u)	Diff	B(u)	Diff	M(u)	Diff
810	10842.5	203	1649.90	970	.47601	199	8.457	25	211.38	145	.0714	4
809	10862.8	204	1659.60	976	.47800	201	8.482	25	212.83	147	.0718	4
808	10883.2	204	1669.36	983	.48001	201	8.507	26	214.30	148	.0722	4
807	10903.6	205	1679.19	989	.48202	202	8.533	25	215.78	149	.0726	4
806	10924.1	205	1689.08	996	.48404	204	8.558	26	217.27	150	.0730	4
805	10944.6	206	1699.04	1003	.48608	204	8.584	26	218.77	151	.0734	4
804	10965.2	206	1709.07	1009	.48812	206	8.610	25	220.28	153	.0738	4
803	10985.8	207	1719.16	1016	.49018	207	8.635	26	221.81	154	.0742	4
802	11006.5	207	1729.32	1023	.49225	207	8.661	26	223.35	155	.0746	4
801	11027.2	208	1739.55	1029	.49432	209	8.687	26	224.90	156	.0750	4
800	11048.0	208	1749.84	1037	.49641	209	8.713	26	226.46	157	.0754	4
799	11068.8	209	1760.21	1043	.49850	211	8.739	26	228.03	158	.0758	4
798	11089.7	210	1770.64	1051	.50061	212	8.765	26	229.61	160	.0762	5
797	11110.7	210	1781.15	1057	.50273	213	8.791	27	231.21	161	.0767	4
796	11131.7	210	1791.72	1065	.50486	214	8.818	26	232.82	163	.0771	4
795	11152.7	211	1802.37	1073	.50700	215	8.844	27	234.45	164	.0775	4
794	11173.8	212	1813.10	1079	.50915	216	8.871	26	236.09	165	.0779	4
793	11195.0	212	1823.89	1087	.51131	217	8.897	27	237.74	167	.0783	5
792	11216.2	213	1834.76	1094	.51348	218	8.924	27	239.41	168	.0788	4
791	11237.5	213	1845.70	1101	.51566	220	8.951	27	241.09	170	.0792	4
790	11258.8	215	1856.71	1116	.51786	222	8.978	27	242.79	171	.0796	4
789	11280.3	215	1867.87	1121	.52008	223	9.005	27	244.50	173	.0800	4
788	11301.8	216	1879.08	1128	.52231	223	9.032	28	246.23	174	.0804	5
787	11323.4	216	1890.36	1134	.52454	224	9.060	27	247.97	175	.0809	4
786	11345.0	216	1901.70	1141	.52678	226	9.087	27	249.72	176	.0813	4
785	11366.6	216	1913.11	1146	.52904	226	9.114	28	251.48	178	.0817	5
784	11388.2	216	1924.57	1153	.53130	227	9.142	28	253.26	178	.0822	4
783	11409.8	217	1936.10	1160	.53357	228	9.170	27	255.04	180	.0826	5
782	11431.5	218	1947.70	1166	.53585	228	9.197	28	256.84	181	.0831	4
781	11453.3	217	1959.36	1172	.53813	230	9.225	28	258.65	182	.0835	5
780	11475.0	218	1971.08	1179	.54043	230	9.253	28	260.47	183	.0840	5
779	11496.8	218	1982.87	1185	.54273	231	9.281	28	262.30	185	.0845	5
778	11518.6	218	1994.72	1192	.54504	232	9.309	28	264.15	186	.0850	4

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
777	11540.4	218	2006.64	1198	.54736	233	9.337	28	266.01	187	.0854	5
776	11562.2	219	2018.62	1206	.54969	234	9.365	29	267.88	189	.0859	5
775	11584.1	219	2030.68	1212	.55203	235	9.394	28	269.77	190	.0864	5
774	11606.0	219	2042.80	1218	.55438	236	9.422	28	271.67	191	.0869	5
773	11627.9	220	2054.98	1226	.55674	237	9.450	29	273.58	192	.0874	4
772	11649.9	220	2067.24	1232	.55911	237	9.479	28	275.50	193	.0878	5
771	11671.9	220	2079.56	1239	.56148	239	9.507	29	277.43	195	.0883	5
770	11693.9	221	2091.95	1246	.56387	239	9.536	29	279.38	196	.0888	5
769	11716.0	220	2104.41	1253	.56626	241	9.565	28	281.34	197	.0893	5
768	11738.0	221	2116.94	1260	.56867	241	9.593	29	283.31	199	.0898	4
767	11760.1	222	2129.54	1267	.57108	242	9.622	29	285.30	200	.0902	5
766	11782.3	222	2142.21	1274	.57350	244	9.651	29	287.30	202	.0907	5
765	11804.5	222	2154.95	1281	.57594	244	9.680	29	289.32	203	.0912	5
764	11826.7	222	2167.76	1288	.57838	245	9.709	29	291.35	204	.0917	5
763	11848.9	222	2180.64	1295	.58083	247	9.738	29	293.39	206	.0922	5
762	11871.1	223	2193.59	1303	.58330	247	9.767	30	295.45	207	.0927	5
761	11893.4	223	2206.62	1309	.58577	248	9.797	29	297.52	208	.0932	5
760	11915.7	223	2219.71	1317	.58825	249	9.826	29	299.60	210	.0937	5
759	11938.0	224	2232.88	1324	.59074	250	9.855	30	301.70	211	.0942	5
758	11960.4	224	2246.12	1332	.59324	251	9.885	29	303.81	213	.0947	5
757	11982.8	225	2259.44	1339	.59575	252	9.914	30	305.94	215	.0952	5
756	12005.3	224	2272.83	1347	.59827	253	9.944	29	308.09	216	.0957	6
755	12027.7	225	2286.30	1354	.60080	254	9.973	30	310.25	217	.0963	5
754	12050.2	226	2299.84	1361	.60334	255	10.003	30	312.42	218	.0968	5
753	12072.8	225	2313.45	1369	.60589	256	10.033	30	314.60	220	.0973	5
752	12095.3	226	2327.14	1377	.60845	258	10.063	30	316.80	222	.0978	6
751	12117.9	226	2340.91	1384	.61103	258	10.093	30	319.02	223	.0984	5
750	12140.5	226	2354.75	1392	.61361	259	10.123	30	321.25	225	.0989	5
749	12163.1	227	2368.67	1399	.61620	260	10.153	31	323.50	226	.0994	6
748	12185.8	227	2382.66	1408	.61880	262	10.184	30	325.76	227	.1000	5
747	12208.5	227	2396.74	1415	.62142	262	10.214	30	328.03	230	.1005	6
746	12231.2	227	2410.89	1423	.62404	263	10.244	31	330.33	231	.1011	5
745	12253.9	228	2425.12	1432	.62667	265	10.275	31	332.64	232	.1016	6

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
744	12276.7	229	2439.44	1439	.62932	266	10.306	30	334.96	234	.1022	5
743	12299.6	228	2453.83	1447	.63198	266	10.336	31	337.30	235	.1027	6
742	12322.4	229	2468.30	1456	.63464	268	10.367	31	339.65	237	.1033	5
741	12345.3	229	2482.86	1463	.63732	269	10.398	31	342.02	239	.1038	6
740	12368.2	229	2497.49	1472	.64001	270	10.429	31	344.41	240	.1044	6
739	12391.1	230	2512.21	1480	.64271	271	10.460	31	346.81	242	.1050	5
738	12414.1	230	2527.01	1488	.64542	272	10.491	31	349.23	244	.1055	6
737	12437.1	230	2541.89	1497	.64814	273	10.522	32	351.67	245	.1061	6
736	12460.1	231	2556.86	1505	.65087	274	10.554	31	354.12	247	.1067	5
735	12483.2	231	2571.91	1513	.65361	276	10.585	31	356.59	249	.1072	6
734	12506.3	231	2587.04	1521	.65637	276	10.616	32	359.08	250	.1078	6
733	12529.4	232	2602.25	1530	.65913	278	10.648	31	361.58	252	.1084	6
732	12552.6	232	2617.55	1539	.66191	279	10.679	32	364.10	254	.1090	6
731	12575.8	232	2632.94	1547	.66470	280	10.711	32	366.64	255	.1096	6
730	12599.0	233	2648.41	1556	.66750	281	10.743	32	369.19	257	.1102	6
729	12622.3	233	2663.97	1561	.67031	282	10.775	32	371.76	259	.1108	6
728	12645.6	233	2679.61	1573	.67313	283	10.807	32	374.35	261	.1114	6
727	12668.9	234	2695.34	1582	.67596	285	10.839	32	376.96	263	.1120	6
726	12692.3	233	2711.16	1591	.67881	286	10.871	32	379.59	264	.1126	7
725	12715.6	234	2727.07	1600	.68167	287	10.903	33	382.23	266	.1133	6
724	12739.0	235	2743.07	1609	.68454	288	10.936	32	384.89	268	.1139	6
723	12762.5	235	2759.16	1617	.68742	289	10.968	33	387.57	269	.1145	6
722	12786.0	235	2775.33	1627	.69031	291	11.001	32	390.26	272	.1151	7
721	12809.5	236	2791.60	1636	.69322	292	11.033	33	392.98	273	.1158	6
720	12833.1	236	2807.96	1645	.69614	293	11.066	33	395.71	275	.1164	6
719	12856.7	236	2824.41	1655	.69907	294	11.099	33	398.46	277	.1170	7
718	12880.3	236	2840.96	1664	.70201	295	11.132	33	401.23	279	.1177	6
717	12903.9	237	2857.60	1673	.70496	297	11.165	33	404.02	281	.1183	6
716	12927.6	237	2874.33	1682	.70793	298	11.198	33	406.83	282	.1189	7
715	12951.3	238	2891.15	1692	.71091	299	11.231	33	409.65	285	.1196	6
714	12975.1	238	2908.07	1701	.71390	301	11.264	33	412.50	287	.1202	7
713	12998.9	238	2925.08	1711	.71691	302	11.297	33	415.37	289	.1209	7
712	13022.7	238	2942.19	1720	.71993	303	11.330	34	418.26	290	.1216	6

TABLE I.—CONTINUED.

u	S(u)	Diff	A(u)	Diff	I(u)	Diff	T (u)	Diff	B(u)	Diff	M(u)	Diff
711	13046.5	239	2959.39	1730	.72296	304	11.364	34	421.16	293	.1222	7
710	13070.4	239	2976.69	1740	.72600	305	11.398	34	424.09	295	.1229	7
709	13094.3	240	2994.09	1749	.72905	307	11.432	33	427.04	297	.1236	7
708	13118.3	240	3011.58	1759	.73212	308	11.465	34	430.01	299	.1243	6
707	13142.3	240	3029.17	1769	.73520	310	11.499	34	433.00	300	.1249	7
706	13166.3	240	3046.86	1780	.73830	311	11.533	34	436.00	303	.1256	7
705	13190.3	241	3064.66	1789	.74141	312	11.567	34	439.03	305	.1263	7
704	13214.4	241	3082.55	1799	.74453	313	11.601	35	442.08	308	.1270	7
703	13238.5	242	3100.54	1810	.74766	315	11.636	34	445.16	309	.1277	7
702	13262.7	242	3118.64	1820	.75081	316	11.670	34	448.25	311	.1284	7
701	13286.9	242	3136.84	1830	.75397	318	11.704	35	451.36	313	.1291	7
700	13311.1	242	3155.14	1841	.75715	319	11.739	35	454.49	315	.1298	7
699	13335.3	243	3173.55	1851	.76034	320	11.774	35	457.64	318	.1305	7
698	13359.6	243	3192.06	1861	.76354	321	11.809	35	460.82	320	.1312	7
697	13383.9	244	3210.67	1872	.76675	323	11.844	35	464.02	322	.1319	7
696	13408.3	244	3229.39	1883	.76998	324	11.879	35	467.24	324	.1326	8
695	13432.7	244	3248.22	1893	.77322	326	11.914	35	470.48	327	.1334	7
694	13457.1	245	3267.15	1904	.77648	327	11.949	35	473.75	329	.1341	7
693	13481.6	245	3286.19	1914	.77975	329	11.984	36	477.04	331	.1348	7
692	13506.1	245	3305.33	1925	.78304	330	12.020	35	480.35	333	.1355	8
691	13530.6	246	3324.58	1937	.78634	332	12.055	36	483.68	336	.1363	7
690	13555.2	246	3343.95	1947	.78966	333	12.091	35	487.04	338	.1370	8
689	13579.8	246	3363.42	1958	.79299	334	12.126	36	490.42	341	.1378	7
688	13604.4	247	3383.00	1970	.79633	336	12.162	36	493.83	343	.1385	8
687	13629.1	247	3402.70	1980	.79969	337	12.198	36	497.26	345	.1393	7
686	13653.8	248	3422.50	1992	.80306	339	12.234	36	500.71	348	.1400	8
685	13678.6	248	3442.42	2003	.80645	340	12.270	36	504.19	350	.1408	8
684	13703.4	248	3462.45	2015	.80985	342	12.306	36	507.69	353	.1416	7
683	13728.2	249	3482.60	2026	.81327	343	12.342	37	511.22	355	.1423	8
682	13753.1	249	3502.86	2038	.81670	345	12.379	36	514.77	357	.1431	8
681	13778.0	249	3523.24	2049	.82015	347	12.415	37	518.34	360	.1439	8
680	13802.9	250	3543.73	2061	.82362	348	12.452	37	521.94	362	.1447	8
679	13827.9	250	3564.34	2074	.82710	349	12.489	37	525.56	365	.1455	8

TABLE 1.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
678	13852.9	250	3585.07	2084	.83059	351	12.526	37	529.21	367	.1463	8
677	13877.9	251	3605 91	2097	.83410	352	12.563	37	532.88	370	.1471	8
676	13903.0	251	3626.88	2108	.83762	354	12.600	37	536.58	373	.1479	8
675	13928.1	252	3647.96	2121	.84116	356	12.637	38	540.31	375	.1487	9
674	13953.3	252	3669.17	2133	.84472	357	12.675	37	544.06	378	.1496	8
673	13978.5	252	3690.50	2144	.84829	359	12.712	38	547.84	381	.1504	8
672	14003.7	253	3711.94	2157	.85188	361	12.750	37	551.65	383	.1512	9
671	14029.0	253	3733.51	2170	.85549	362	12.787	38	555.48	386	.1521	8
670	14054.3	253	3755.21	2182	.85911	363	12.825	38	559.34	389	.1529	8
669	14079.6	254	3777.03	2195	.86274	365	12.863	38	563.23	391	.1537	9
668	14105.0	254	3798.98	2207	.86639	367	12.901	38	567.14	394	.1546	9
667	14130.4	255	3821.05	2219	.87006	369	12.939	38	571.08	397	.1555	8
666	14155.9	255	3843.24	2233	.87375	370	12.977	38	575.05	400	.1563	9
665	14181.4	255	3865.57	2245	.87745	372	13.015	38	579.05	403	.1572	9
664	14206.9	256	3888.02	2258	.88117	373	13.053	39	583.08	405	.1581	8
663	14232.5	256	3910.60	2271	.88490	376	13.092	38	587.13	408	.1589	9
662	14258.1	256	3933.31	2285	.88866	377	13.130	39	591.21	411	.1598	9
661	14283.7	257	3956.16	2297	.89243	379	13.169	39	595.32	414	.1607	9
660	14309.4	257	3979.13	2311	.89622	380	13.208	39	599.46	417	.1616	9
659	14335.1	258	4002.24	2324	.90002	382	13.247	39	603.63	419	.1625	9
658	14360.9	258	4025.48	2338	.90384	384	13.286	40	607.82	423	.1634	9
657	14386.7	259	4048.86	2351	.90768	385	13.326	39	612.05	426	.1643	9
656	14412.6	259	4072.37	2364	.91153	388	13.365	39	616.31	429	.1652	9
655	14438.5	259	4096.01	2378	.91541	389	13.404	40	620.60	432	.1661	10
654	14464.4	260	4119.79	2392	.91930	391	13.444	40	624.92	435	.1671	9
653	14490.4	260	4143.71	2406	.92321	394	13.484	40	629.27	438	.1680	9
652	14516.4	260	4167.77	2419	.92715	395	13.524	40	633.65	441	.1689	10
651	14542.4	261	4191.96	2434	.93110	396	13.564	40	638.06	444	.1699	9
650	14568.5	261	4216.30	2448	.93506	398	13.604	40	642.50	447	.1708	10
649	14594.6	262	4240.78	2462	.93904	400	13.644	40	646.97	451	.1718	9
648	14620.8	262	4265.40	2476	.94304	402	13.684	41	651.48	454	.1727	10
647	14647.0	262	4290.16	2491	.94706	404	13.725	41	656.02	458	.1737	9
646	14673 2	263	4315.07	2505	.95110	406	13.766	40	660.60	460	.1746	10

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$J(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
645	14699.5	264	4340.12	2520	.95516	407	13.806	41	665.20	464	.1756	10
644	14725.9	264	4365.32	2535	.95923	410	13.847	41	669.84	467	.1766	10
643	14752.3	264	4390.67	2549	.96333	412	13.888	41	674.51	470	.1776	10
642	14778.7	264	4416.16	2565	.96745	413	13.929	42	679.21	474	.1786	10
641	14805.1	265	4441.81	2579	.97158	416	13.971	41	683.95	477	.1796	10
640	14831.6	265	4467.60	2595	.97574	417	14.012	41	688.72	480	.1806	10
639	14858.1	266	4493.55	2609	.97991	419	14.053	42	693.52	484	.1816	11
638	14884.7	266	4519.64	2625	.98410	421	14.095	42	698.36	488	.1827	10
637	14911.3	267	4545.89	2641	.98831	423	14.137	42	703.24	491	.1837	10
636	14938.0	267	4572.30	2659	.99254	426	14.179	42	708.15	494	.1847	11
635	14964.7	267	4598.86	2671	.99680	427	14.221	42	713.09	498	.1858	10
634	14991.4	268	4625.57	2687	1.00107	429	14.263	42	718.07	502	.1868	11
633	15018.2	268	4652.44	2703	1.00536	431	14.305	43	723.09	506	.1879	11
632	15045.0	269	4679.47	2718	1.00967	434	14.348	42	728.15	509	.1890	10
631	15071.9	269	4706.65	2735	·.01401	436	14.390	43	733.24	512	.1900	11
630	15098.8	270	4734.00	2751	1.01837	437	14.433	43	738.36	516	.1911	11
629	15125.8	270	4761.51	2767	1.02274	439	14.476	43	743.52	521	.1922	11
628	15152.8	270	4789.18	2784	1.02713	442	14.519	43	748.73	524	.1933	11
627	15179.8	271	4817.02	2800	1.03155	443	14.562	43	753.97	528	.1944	11
626	15206.9	271	4845.02	2816	1.03598	446	14.605	43	759.25	532	.1955	11
625	15234.c	272	4873.18	2833	1.04044	448	14 648	44	764.57	535	.1966	11
624	15261.2	272	4901.51	2849	1.04492	451	14.692	43	769.92	540	.1977	11
623	15288.4	273	4930.00	2867	1.04943	452	14.735	44	775.32	543	.1988	11
622	15315.7	273	4958.67	2883	1.05395	455	14.779	44	780.75	547	.1999	12
621	15343.0	273	4987.50	2901	1.05850	457	14.823	44	786.22	551	.2011	11
620	15370.3	274	5016.51	2918	1.06307	459	14.867	44	791.73	555	.2022	11
619	15397.7	274	5045.69	2935	1.06766	461	14.911	45	797.28	56c	.2033	12
618	15425.1	275	5075.04	2953	1.07227	463	14.956	44	802.88	563	.2045	12
617	15452.6	275	5104.57	2970	1.07690	466	15.000	45	808.51	568	.2057	11
616	15480.1	276	5134.27	2988	1.08156	468	15.045	45	814.19	572	.2068	12
615	15507.7	276	5164.15	3006	1.08624	471	15.090	45	819.91	576	.2080	12
614	15535.3	277	5194.21	3023	1.09095	473	15.135	45	825.67	580	.2092	12
613	15563.0	277	5224.44	3042	1.09568	475	15.180	45	831.47	585	.2104	12

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
612	15590.7	277	5254.86	3060	1.10043	477	15.225	45	837.32	588	.2116	12
611	15618.4	278	5285.46	3078	1.10520	480	15.270	46	843.20	593	.2128	12
610	15646.2	278	5316.24	3097	1.11000	482	15.316	45	849.13	597	.2140	12
609	15674.0	279	5347.21	3115	1.11482	484	15.361	46	855.10	602	.2152	13
608	15701.9	279	5378.36	3135	1.11966	486	15.407	46	861.12	607	.2165	12
607	15729.8	280	5409.71	3153	1.12452	489	15.453	46	867.19	611	.2177	13
606	15757.8	280	5441.24	3171	1.12941	492	15.499	47	873.30	615	.2190	13
605	15785.8	281	5472.95	3191	1.13433	494	15.546	46	879.45	620	.2203	12
604	15813.9	281	5504.86	3210	1.13927	497	15.592	46	885.65	625	.2215	13
603	15842.0	281	5536.96	3230	1.14424	499	15.638	47	891.90	630	.2228	13
602	15870.1	282	5569.26	3249	1.14923	502	15.685	47	898.20	634	.2241	13
601	15898.3	283	5601.75	3268	1.15425	504	15.732	47	904.54	638	.2254	13
600	15926.6	283	5634.43	3288	1.15929	506	15.779	47	910.92	643	.2267	13
599	15954.9	283	5667.31	3309	1.16435	509	15.826	47	917.35	649	.2280	13
598	15983.2	284	5700.40	3329	1.16944	512	15.873	48	923.84	653	.2293	14
597	16011.6	285	5733.69	3349	1.17456	514	15.921	47	930.37	658	.2307	13
596	16040.1	285	5767.18	3369	1.17970	517	15.968	48	936.95	663	.2320	14
595	16068.6	285	5800.87	3389	1.18487	519	16.016	48	943.58	668	.2334	13
594	16097.1	286	5834.76	3409	1.19006	522	16.064	49	950.26	674	.2347	14
593	16125.7	287	5868.85	3431	1.19528	525	16.113	48	957.00	678	.2361	13
592	16154.4	287	5903.16	3451	1.20053	527	16.161	48	963.78	683	2374	14
591	16183.1	287	5937.67	3472	1.20580	530	16.209	49	970.61	688	.2388	14
590	16211.8	288	5972.39	3493	1.21110	533	16.258	49	977.49	693	.2402	14
589	16240.6	288	6007.32	3515	1.21643	535	16.307	49	984.42	699	.2416	14
588	16269.4	289	6042.47	3536	1.22178	538	16.356	49	991.41	705	.2430	14
587	16298.3	289	6077.83	3558	1.22716	541	16.405	49	998.46	709	.2444	15
586	16327.2	290	6113.41	3579	1.23257	544	16.454	50	1005.6	71	.2459	14
585	16356.2	290	6149.20	3602	1.23801	547	16.504	49	1012.7	72	.2473	15
584	16385.2	291	6185.22	3624	1.24348	549	16.553	50	1019.9	73	.2488	14
583	16414.3	291	6221.46	3646	1.24897	552	16.603	50	1027.2	73	.2502	15
582	16443.4	292	6257.92	3669	1.25449	555	16.653	51	1034.5	73	.2517	15
581	16472.6	292	6294.61	3691	1.26004	558	16.704	50	1041.8	74	.2532	15
580	16501.8	293	6331.52	3714	1.26562	561	16.754	51	1049.2	75	2547	15

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
579	16531.1	293	6368.66	3735	1.27123	564	16.805	50	1056.7	76	.2562	16
578	16560.4	294	6406.01	3762	1.27687	566	16.855	51	1064.3	76	.2578	15
577	16589.8	294	6443.63	3783	1.28253	570	16.906	52	1071.9	76	.2593	15
576	16619.2	295	6481.46	3806	1.28823	573	16.958	51	1079.5	77	.2608	16
575	16648.7	295	6519.52	3830	1.29396	575	17.009	51	1087.2	78	.2624	16
574	16678.2	296	6557.82	3854	1.29971	579	17.060	52	1095.0	78	.2640	15
573	16707.8	296	6596.36	3878	1.30550	581	17.112	52	1102.8	79	.2655	16
572	16737.4	297	6635.14	3902	1.31131	585	17.164	52	1110.7	80	.2671	16
571	16767.1	298	6674.16	3926	1.31716	588	17.216	52	1118.7	80	.2687	16
570	16796.9	298	6713.42	3951	1.32304	591	17.268	52	1126.7	81	.2703	16
569	16826.7	299	6752.93	3975	1.32895	594	17.320	53	1134.8	81	.2719	16
568	16856.6	299	6792.68	4000	1.33489	597	17.373	52	1142.9	82	.2735	16
567	16886.5	299	6832.68	4025	1.34086	600	17.425	53	1151.1	83	.2751	17
566	16916.4	300	6872.93	4050	1.34686	604	17.478	53	1159.4	83	.2768	16
565	16946.4	301	6913.43	4075	1.35290	607	17.531	53	1167.7	84	.2784	17
564	16976.5	301	6954.18	4101	1.35897	610	17.584	54	1176.1	85	.2801	17
563	17006.6	302	6995.19	4127	1.36507	613	17.638	53	1184.6	85	.2818	17
562	17036.8	302	7036.46	4153	1.37120	616	17.691	54	1193.1	86	.2835	17
561	17067.0	303	7077.99	4179	1.37736	620	17.745	54	1201.7	86	.2852	17
560	17097.3	303	7119.78	4205	1.38356	623	17.799	54	1210.3	88	.2869	17
559	17127.6	304	7161.83	4232	1.38979	627	17.853	55	1219.1	88	.2886	18
558	17158.0	304	7204.15	4258	1.39606	630	17.908	54	1227.9	88	.2904	17
557	17188.4	305	7246.73	4285	1.40236	633	17.962	55	1236.7	90	.2921	18
556	17218.9	305	7289.58	4313	1.40869	637	18.017	55	1245.7	90	.2939	18
555	17249.4	306	7332.71	4340	1.41506	640	18.072	55	1254.7	91	.2957	18
554	17280.0	307	7376.11	4367	1.42146	643	18.127	56	1263.8	91	.2975	18
553	17310.7	307	7419.78	4396	1.42789	647	18.183	55	1272.9	92	.2993	18
552	17341.4	308	7463.74	4423	1.43436	651	18.238	56	1282.1	93	.3011	19
551	17372.2	308	7507.97	4451	1.44087	654	18.294	56	1291.4	94	.3030	18
550	17403.0	309	7552.48	4480	1.44741	658	18.350	56	1300.8	94	.3048	19
549	17433.9	309	7597.28	4508	1.45199	661	18.406	56	1310.2	95	.3067	18
548	17464.8	310	7642.36	4537	1.46060	665	18.462	57	1319.7	96	.3085	19
547	17495.8	310	7687.73	4566	1.46725	669	18.519	57	1329.3	97	.3104	19

TABLE 1.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
546	17526.8	311	7733.39	4595	1.47394	672	18.576	57	1339.0	97	.3123	19
545	17557.9	312	7779.34	4624	1.48066	676	18.633	57	1348.7	99	.3142	19
544	17589.1	312	7825.58	4654	1.48742	680	18.690	57	1358.6	99	.3161	20
543	17620.3	313	7872.12	4684	1.49422	684	18.747	58	1368.5	99	.3181	19
542	17651.6	313	7918.96	4716	1.50106	687	18.805	58	1378.4	101	.3200	20
541	17682.9	314	7966.12	4743	1.50793	691	18.863	58	1388.5	101	.3220	20
540	17714.3	315	8013.55	4775	1.51484	695	18.921	58	1398.6	103	.3240	20
539	17745.8	315	8061.30	4806	1.52179	699	18.979	59	1408.9	103	.3260	20
538	17777.3	316	8109.36	4837	1.52878	703	19.038	58	1419.2	103	.3280	21
537	17808.9	316	8157.73	4868	1.53581	706	19.096	59	1429.5	105	.3301	20
536	17840.5	317	8206.41	4900	1.54287	711	19.155	60	1440.0	106	.3321	21
535	17872.2	317	8255.41	4932	1.54998	715	19.215	59	1450.6	106	.3342	21
534	17903.9	318	8304.73	4963	1.55713	718	19.274	60	1461.2	108	.3363	21
533	17935.7	319	8354.36	4996	1.56431	723	19.334	60	1472.0	108	.3384	21
532	17967.6	319	8404.32	5029	1.57154	727	19.394	60	1482.8	109	.3405	22
531	17999.5	320	8454.61	5061	1.57881	731	19.454	60	1493.7	110	.3427	21
530	18031.5	320	8505.22	5094	1.58612	735	19.514	60	1504.7	110	.3448	22
529	18063.5	321	8556.16	5128	1.59347	739	19.574	61	1515.7	112	.3470	21
528	18095.6	322	8607.44	5162	1.60086	744	19.635	61	1526.9	113	.3491	22
527	18127.8	322	8659.06	5195	1.60830	748	19.696	61	1538.2	113	.3513	22
526	18160.0	323	8711.01	5229	1.61578	752	19.757	62	1549.5	115	.3535	22
525	18192.3	324	8763.30	5264	1.62330	756	19.819	62	1561.0	115	.3557	23
524	18224.7	324	8815.94	5298	1.63086	761	19.881	62	1572.5	117	.3580	22
523	18257.1	325	8868.92	5333	1.63847	765	19.943	62	1584.2	117	.3602	23
522	18289.6	325	8922.25	5368	1.64612	769	20.005	62	1595.9	118	.3625	23
521	18322.1	326	8975.93	5404	1.65381	774	20.067	63	1607.7	120	.3648	23
520	18354.7	327	9029.97	5439	1.66155	778	20.130	63	1619.7	120	.3671	23
519	18387.4	327	9084.36	5475	1.66933	783	20.193	63	1631.7	121	.3694	24
518	18420.1	328	9139.11	5512	1.67716	788	20.256	63	1643.8	122	.3718	23
517	18452.9	328	9194.23	5548	1.68504	792	20.319	64	1656.0	124	.3741	24
516	18485.7	329	9249.71	5585	1.69296	796	20.383	64	1668.4	124	.3765	24
515	18518.6	330	9305.56	5623	1.70092	802	20.447	64	1680.8	126	.3789	24
514	18551.6	331	9361.79	5660	1.70894	806	20.511	64	1693.4	126	.3813	25

TABLE I.—Continued.

u	S(u)	Diff	A(u)	Diff	I(u)	Diff	T(u)	Diff	B(u)	Diff	M(u)	Diff
513	18584.7	331	9418.39	5699	1.71700	810	20.575	65	1706.0	127	.3838	24
512	18617.8	332	9475.38	5736	1.72510	816	20.640	65	1718.7	129	.3862	25
511	18651.0	332	9532.74	5775	1.73326	820	20.705	65	1731.6	130	.3887	25
510	18684.2	333	9590.49	5813	1.74146	825	20.770	65	1744.6	131	.3912	25
509	18717.5	334	9648.62	5853	1.74971	830	20.835	66	1757.7	132	.3937	25
508	18750.9	334	9707.15	5891	1.75801	835	20.901	66	1770.9	133	.3962	26
507	18784.3	335	9766.06	5932	1.76636	840	20.967	66	1784.2	134	.3988	26
506	18817.8	336	9825.38	5971	1.77476	845	21.033	66	1797.6	135	.4014	26
505	18851.4	336	9885.09	6012	1.78321	850	21.099	67	1811.1	136	.4040	26
504	18885.0	337	9945.21	6053	1.79171	855	21.166	67	1824.7	138	.4066	26
503	18918.7	338	10005.74	6093	1.80026	860	21.233	67	1838.5	138	.4092	27
502	18952.5	338	10066.67	6134	1.80886	865	21.300	67	1852.3	140	.4119	27
501	18986.3	339	10128.01	6177	1.81751	871	21.367	68	1866.3	141	.4146	27
500	19020.2	340	10189.78	6219	1.82622	876	21.435	68	1880.4	142	.4173	27
499	19054.2	340	10251.9	626	1.83498	881	21.503	69	1894.6	143	.4200	28
498	19088.2	341	10314.5	631	1.84379	886	21.572	69	1908.9	145	.4228	27
497	19122.3	341	10377.6	634	1.85265	892	21.641	69	1923.4	146	.4255	28
496	19156.4	342	10441.0	639	1.86157	897	21.710	69	1938.0	147	.4283	29
495	19190.6	343	10504.9	644	1.87054	903	21.779	69	1952.7	148	.4312	28
494	19224.9	344	10569.3	648	1.87957	908	21.848	70	1967.5	150	.4340	29
493	19259.3	345	10634.1	652	1.88865	913	21.918	70	1982.5	151	.4369	29
492	19293.8	345	10699.3	657	1.89778	919	21.988	70	1997.6	152	.4398	29
491	19328.3	346	10765.0	661	1.90697	925	22.058	70	2012.8	154	.4427	29
490	19362.9	347	10831.1	665	1.91622	930	22.128	71	2028.2	155	.4456	30
489	19397.6	347	10897.6	671	1.92552	936	22.199	71	2043.7	156	.4486	29
488	19432.3	348	10964.7	675	1.93488	942	22.270	71	2059.3	158	.4515	30
487	19467.1	349	11032.2	679	1.94430	948	22.341	72	2075.1	159	.4545	31
486	19502.0	349	11100.1	685	1.95378	954	22.413	72	2091.0	161	.4576	30
485	19536.9	351	11168.6	689	1.96332	960	22.485	72	2107.1	162	.4606	31
484	19572.0	351	11237.5	695	1.97292	966	22.557	73	2123.3	163	.4637	31
483	19607.1	351	11307.0	699	1.98258	972	22.630	73	2139.6	164	.4668	32
482	19642.2	353	11376.9	703	1.99230	977	22.703	73	2156.0	166	.4700	31
481	19677.5	353	11447.2	709	2.00207	983	22.776	73	2172.6	168	.4731	32

TABLE I.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
480	19712.8	354	11518.1	713	2.01190	990	22.849	74	2189.4	169	.4763	32
479	19748.2	354	11589.4	719	2.02180	996	22.923	74	2206.3	171	.4795	32
478	19783.6	355	11661.3	724	2.03176	1003	22.997	74	2223.4	172	.4827	33
477	19819.1	356	11733.7	729	2.04179	1009	23.071	75	2240.6	173	.4860	33
476	19854.7	357	11806.6	734	2.05188	1015	23.146	75	2257.9	175	.4893	33
475	19890.4	358	11880.0	739	2.06203	1022	23.221	75	2275.4	177	.4926	33
474	19926.2	358	11953.9	745	2.07225	1028	23.296	76	2293.1	179	.4959	34
473	19962.0	359	12028.4	750	2.08253	1035	23.372	76	2311.0	180	.4993	34
472	19997.9	360	12103.4	755	2.09288	1041	23.448	76	2329.0	181	.5027	34
471	20033.9	361	12178.9	760	2.10329	1047	23.524	77	2347.1	183	.5061	35
470	20070.0	362	12254.9	766	2.11376	1054	23.601	77	2365.4	185	.5096	35
469	20106.2	362	12331.5	771	2.12430	1061	23.678	77	2383.9	187	.5131	35
468	20142.4	363	12408.6	777	2.13491	1068	23.755	78	2402.6	188	.5166	36
467	20178.7	363	12486.3	783	2.14559	1076	23.833	78	2421.4	190	.5202	35
466	20215.0	365	12564.6	788	2.15635	1082	23.911	78	2440.4	192	.5237	36
465	20251.5	365	12643.4	794	2.16717	1089	23.989	79	2459.6	194	.5273	37
464	20288.0	367	12722.8	799	2.17806	1096	24.068	79	2479.0	195	.5310	37
463	20324.7	367	12802.7	805	2.18902	1104	24.147	79	2498.5	197	.5347	37
462	20361.4	367	12883.2	811	2.20006	1110	24.226	80	2518.2	198	.5384	37
461	20398.1	369	12964.3	816	2.21116	1117	24.306	80	2538.0	200	.5421	38
460	20435.0	369	13045.9	822	2.22233	1124	24.386	80	2558.0	202	.5459	38
459	20471.9	370	13128.1	829	2.23357	1132	24.466	81	2578.2	204	.5497	38
458	20508.9	371	13211.0	834	2.24489	1140	24.547	81	2598.6	206	.5535	39
457	20546.0	371	13294.4	841	2.25629	1147	24.628	82	2619.2	208	.5574	39
456	20583.1	373	13378.5	848	2.26776	1155	24.710	82	2640.0	210	.5613	40
455	20620.4	373	13463.3	853	2.27931	1163	24.792	82	2661.0	212	.5653	39
454	20657.7	374	13548.6	859	2.29094	1171	24.874	82	2682.2	213	.5692	40
453	20695.1	375	13634.5	866	2.30265	1178	24.956	83	2703.5	216	.5732	41
452	20732.6	376	13721.1	872	2.31443	1185	25.039	83	2725.1	218	.5773	41
451	20770.2	377	13808.3	878	2.32628	1193	25.122	84	2746.9	219	.5814	41
450	20807.9	377	13896.1	885	2.33821	1201	25.206	84	2768.8	222	.5855	41
449	20845.6	378	13984.6	891	2.35022	1210	25.290	84	2791.0	224	.5896	42
448	20883.4	380	14073.7	898	2.36232	1218	25.374	85	2813.4	226	.5938	43

TABLE 1.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
447	20921.4	380	14163.5	905	2.37450	1226	25.459	85	2836.0	228	.5981	42
446	20959.4	380	14254.0	911	2.38676	1235	25.544	85	2858.8	231	.6023	43
445	20997.4	382	14345.1	919	2.39911	1243	25.629	86	2881.9	232	.6066	44
444	21035.6	383	14437.0	925	2.41154	1251	25.715	86	2905.1	235	.6110	44
443	21073.9	383	14529.5	932	2.42405	1260	25.801	87	2928.6	237	.6154	44
442	21112.2	385	14622.7	939	2.43665	1268	25.888	87	2952.3	239	.6198	45
441	21150.7	385	14716.6	946	2.44933	1276	25.975	87	2976.2	241	.6243	45
440	21189.2	386	14811.2	953	2.46209	1285	26.062	88	3000.3	243	.6288	45
439	21227.8	387	14906.5	960	2.47494	1294	26.150	88	3024.6	246	.6333	46
438	21266.5	388	15002.5	968	2.48788	1303	26.238	89	3049.2	249	.6379	46
437	21305.3	389	15099.3	975	2.50091	1313	26.327	89	3074.1	250	.6425	47
436	21344.2	389	15196.8	982	2.51404	1322	26.416	89	3099.1	253	.6472	47
435	21383.1	391	15295.0	990	2.52726	1331	26.505	90	3124.4	256	.6519	48
434	21422.2	392	15394.0	997	2.54057	1340	26.595	90	3150.0	258	.6567	48
433	21461.4	392	15493.7	1005	2.55397	1349	26.685	91	3175.8	261	.6615	48
432	21500.6	394	15594.2	1012	2.56746	1358	26.776	91	3201.9	263	.6663	49
431	21540.0	394	15695.4	1019	2.58104	1367	26.867	92	3228.2	26[illegible]	.6712	50
430	21579.4	395	15797.3	1027	2.59471	1377	26.959	92	3254.7	268	.6762	50
429	21618.9	396	15900.0	1035	2.60848	1387	27.051	92	3281.5	271	.6812	50
428	21658.5	397	16003.5	1044	2.62235	1397	27.143	93	3308.6	273	.6862	51
427	21698.2	398	16107.9	1052	2.63632	1407	27.236	93	3335.9	277	.6913	52
426	21738.0	399	16213.1	1060	2.65039	1417	27.329	94	3363.6	279	.6965	52
425	21777.9	399	16319.1	1068	2.66456	1427	27.423	94	3391.5	281	.7017	52
424	21817.8	401	16425.9	1076	2.67883	1437	27.517	95	3419.6	285	.7069	53
423	21857.9	402	16533.5	1084	2.69320	1447	27.612	95	3448.1	287	.7122	53
422	21898.1	403	16641.9	1093	2.70767	1458	27.707	96	3476.8	290	.7175	54
421	21938.4	403	16751.2	1101	2.72225	1467	27.803	96	3505.8	292	.7229	54
420	21978.7	404	16861.3	1109	2.73692	1477	27.899	96	3535.0	296	.7283	55
419	22019.1	405	16972.2	1119	2.75169	148[illegible]	27.995	97	3564.6	298	.7338	55
418	22059.6	406	17084.1	1127	2.76658	150[illegible]	28.092	97	3594.4	302	.7393	56
417	22100.2	407	17196.8	1137	2.78158	151[illegible]	28.189	98	3624.6	305	.7449	56
416	22140.9	409	17310.5	1145	2.79668	152[illegible]	28.287	98	3655.1	307	.7505	57
415	22181.8	409	17425.0	1155	2.81190	153[illegible]	28.385	99	3685.8	311	.7562	57

TABLE I.—Continued.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff	$B(u)$	Diff	$M(u)$	Diff
414	22222.7	410	17540.5	1163	2.82723	1544	28.484	99	3716.9	314	.7619	58
413	22263.7	411	17656.8	1173	2.84267	1555	28.583	100	3748.3	317	.7677	59
412	22304.8	413	17774.1	1181	2.85822	1566	28.683	100	3780.0	320	.7736	59
411	22346.1	413	17892.2	1191	2.87388	1577	28.783	101	3812.0	323	.7795	60
410	22387.4	414	18011.3	1200	2.88965	1589	28.884	101	3844.3	326	.7855	60
409	22428.8	416	18131.3	1211	2.90554	1601	28.985	102	3876.9	330	.7915	61
408	22470.4	416	18252.4	1220	2.92155	1613	29.087	102	3909.9	334	.7976	61
407	22512.0	417	18374.4	1230	2.93768	1625	29.189	103	3943.3	337	.8037	62
406	22553.7	419	18497.4	1240	2.95393	1637	29.292	103	3977.0	340	.8099	63
405	22595.6	419	18621.4	1250	2.97030	1649	29.395	104	4011.0	344	.8162	64
404	22637.5	421	18746.4	1259	2.98679	1662	29.499	104	4045.4	347	.8226	64
403	22679.6	422	18872.3	1270	3.00341	1674	29.603	105	4080.1	351	.8290	65
402	22721.8	422	18999.3	1280	3.02015	1686	29.708	105	4115.2	354	.8355	65
401	22764.0	424	19127.3	1289	3.03701	1698	29.813	106	4150.6	358	.8420	66
400	22806.4	424	19256.2	1300	3.05399	1710	29.919	106	4186.4	361	.8486	66

TABLE I.—CONTINUED.—*Auxiliary A.*

z	1200	Δ_z	Δ_v	1250	Δ_z	Δ_v	1300	Δ_z	Δ_v	1350	Δ_z	Δ_v	1400	Δ_z	Δ_v
400	.0092	24	7	.0085	23	6	.0079	21	6	.0073	20	5	.0068	18	5
500	.0116	24	8	.0108	22	8	.0100	21	7	.0093	19	7	.0086	18	6
600	.0140	25	10	.0130	23	9	.0121	21	9	.0112	20	8	.0104	19	7
700	.0165	26	12	.0153	24	11	.0142	22	10	.0132	20	9	.0123	19	8
800	.0191	26	14	.0177	24	13	.0164	22	12	.0152	21	10	.0142	19	9
900	.0217	26	16	.0201	24	15	.0186	23	13	.0173	21	12	.0161	20	11
1000	.0243	27	18	.0225	25	16	.0209	23	15	.0194	22	13	.0181	20	12
1100	.0270	27	20	.0250	25	18	.0232	23	16	.0216	23	15	.0201	20	14
1200	.0297	27	22	.0275	26	20	.0255	24	18	.0237	23	16	.0221	21	15
1300	.0324	28	23	.0301	26	22	.0279	24	19	.0260	22	18	.0242	21	17
1400	.0352	28	25	.0327	26	24	.0303	24	21	.0282	23	19	.0263	22	19
1500	.0380	28	27	.0353	27	26	.0327	25	22	.0305	23	20	.0285	22	20
1600	.0408	28	28	.0380	27	28	.0352	26	24	.0328	24	21	.0307	22	21
1700	.0436	29	29	.0407	27	29	.0378	26	26	.0352	25	24	.0329	23	22
1800	.0465	30	31	.0434	28	30	.0404	26	27	.0377	25	25	.0352	23	24
1900	.0495	30	33	.0462	28	32	.0430	27	28	.0402	25	27	.0375	23	25
2000	.0525	30	35	.0490	28	33	.0457	27	30	.0427	25	29	.0398	24	26
2100	.0555	31	37	.0518	29	34	.0484	27	32	.0452	26	30	.0422	24	27
2200	.0586	31	39	.0547	29	36	.0511	27	33	.0478	26	32	.0446	25	28
2300	.0617	31	41	.0576	30	38	.0538	28	34	.0504	27	33	.0471	26	30
2400	.0648	32	42	.0606	30	40	.0566	29	35	.0531	27	34	.0497	25	32
2500	.0680	32	44	.0636	30	41	.0595	28	37	.0558	27	36	.0522	26	33
2600	.0712	32	46	.0666	31	43	.0623	29	38	.0585	27	37	.0548	26	34
2700	.0744	33	47	.0697	31	45	.0652	30	40	.0612	28	38	.0574	26	36
2800	.0777	33	49	.0728	32	46	.0682	30	42	.0640	29	40	.0600	27	37
2900	.0810	33	50	.0760	31	48	.0712	30	43	.0669	29	42	.0627	28	38
3000	.0843	33	52	.0791	32	49	.0742	30	44	.0698	29	43	.0655	28	40
3100	.0876	34	53	.0823	32	51	.0772	31	45	.0727	29	44	.0683	28	41
3200	.0910	34	55	.0855	33	52	.0803	32	47	.0756	30	45	.0711	28	43
3300	.0944	35	56	.0888	33	53	.0835	32	49	.0786	30	47	.0739	29	44
3400	.0979	35	58	.0921	34	56	.0867	33	51	.0816	31	48	.0768	29	45
3500	.1014	35	59	.0955	34	56	.0900	32	53	.0847	31	51	.0797	29	47
3600	.1049	36	60	.0989	34	57	.0932	32	54	.0878	31	52	.0826	30	48

TABLE 1.—CONTINUED.—*Auxiliary A.*

z	1450	Δ_z	Δ_v	1500	Δ_z	Δ_v	1550	Δ_z	Δ_v	1600	Δ_z	Δ_v	1650	Δ_z	Δ_v
400	.0063	17	3	.0060	15	4	.0056	14	4	.0052	14	3	.0049	13	3
500	.0080	17	5	.0075	16	5	.0070	14	4	.0066	14	4	.0062	13	4
600	.0097	18	6	.0091	16	6	.0085	15	5	.0080	14	5	.0075	14	4
700	.0115	18	8	.0107	17	7	.0100	16	6	.0094	15	5	.0089	14	5
800	.0133	18	9	.0124	17	8	.0116	16	7	.0109	15	6	.0103	14	6
900	.0151	18	10	.0141	17	9	.0132	16	8	.0124	15	7	.0117	14	7
1000	.0169	18	11	.0158	17	10	.0148	16	9	.0139	15	8	.0131	14	8
1100	.0187	19	12	.0175	18	11	.0164	17	10	.0154	16	9	.0145	15	9
1200	.0206	19	13	.0193	17	12	.0181	17	11	.0170	16	10	.0160	14	10
1300	.0225	19	15	.0210	18	12	.0198	17	12	.0186	16	12	.0174	16	10
1400	.0244	21	16	.0228	19	13	.0215	17	13	.0202	16	12	.0190	16	11
1500	.0265	21	18	.0247	19	15	.0232	17	14	.0218	17	12	.0206	16	12
1600	.0286	21	20	.0266	20	17	.0249	19	14	.0235	17	13	.0222	16	13
1700	.0307	21	21	.0286	20	18	.0268	19	16	.0252	18	14	.0238	16	14
1800	.0328	22	22	.0306	21	19	.0287	20	17	.0270	18	16	.0254	17	15
1900	.0350	22	23	.0327	21	20	.0307	19	19	.0288	18	17	.0271	17	16
2000	.0372	23	24	.0348	21	22	.0326	20	20	.0306	19	18	.0288	18	17
2100	.0395	23	26	.0369	22	23	.0346	21	21	.0325	19	19	.0306	18	18
2200	.0418	23	27	.0391	22	24	.0367	20	23	.0344	20	20	.0324	18	19
2300	.0441	24	28	.0413	22	26	.0387	21	23	.0364	20	22	.0342	19	20
2400	.0465	24	30	.0435	23	27	.0408	22	24	.0384	20	23	.0361	19	21
2500	.0489	25	31	.0458	23	28	.0430	22	26	.0404	20	24	.0380	19	22
2600	.0514	24	33	.0481	23	29	.0452	22	28	.0424	21	25	.0399	20	23
2700	.0538	25	34	.0504	24	30	.0474	22	29	.0445	21	26	.0419	20	24
2800	.0563	26	35	.0528	24	32	.0496	23	30	.0466	22	27	.0439	21	25
2900	.0589	26	37	.0552	25	33	.0519	23	31	.0488	22	28	.0460	21	26
3000	.0615	27	38	.0577	25	35	.0542	24	33	.0510	22	29	.0481	21	27
3100	.0642	26	40	.0602	25	36	.0566	24	34	.0532	23	30	.0502	22	28
3200	.0668	27	41	.0627	26	37	.0590	25	35	.0555	23	31	.0524	22	30
3300	.0695	28	42	.0653	26	38	.0615	24	37	.0578	23	32	.0546	22	31
3400	.0723	27	44	.0679	26	40	.0639	25	38	.0601	24	33	.0568	22	32
3500	.0750	28	45	.0705	27	41	.0664	25	39	.0625	24	35	.0590	23	33
3600	.0778	28	46	.0732	27	43	.0689	26	40	.0649	23	36	.0613	23	34

TABLE I.—Continued.—*Auxiliary A.*

z	1700	Δ_s	Δ_v	1750	Δ_s	Δ_v	1800	Δ_s	Δ_v	1850	Δ_s	Δ_v	1900	Δ_s	Δ_v
400	.0046	12	3	.0043	12	2	.0041	11	2	.0039	10	2	.0037	10	2
500	.0058	13	3	.0055	12	3	.0052	11	3	.0049	11	2	.0047	10	2
600	.0071	13	4	.0067	12	4	.0063	11	3	.0060	10	3	.0057	10	3
700	.0084	13	5	.0079	12	5	.0074	12	4	.0070	11	3	.0067	10	4
800	.0097	13	6	.0091	12	5	.0086	11	5	.0081	11	4	.0077	10	4
900	.0110	13	7	.0103	12	6	.0097	12	5	.0092	11	5	.0087	11	4
1000	.0123	13	8	.0115	13	6	.0109	12	6	.0103	12	5	.0098	11	5
1100	.0136	14	8	.0128	14	7	.0121	13	6	.0115	12	6	.0109	11	5
1200	.0150	14	8	.0142	13	8	.0134	12	7	.0127	12	7	.0120	11	6
1300	.0164	15	9	.0155	14	9	.0146	13	7	.0139	12	8	.0131	12	6
1400	.0179	15	10	.0169	13	10	.0159	13	8	.0151	12	8	.0143	12	7
1500	.0194	15	12	.0182	14	10	.0172	14	9	.0163	13	8	.0155	12	8
1600	.0209	15	13	.0196	15	10	.0186	13	10	.0176	13	9	.0167	12	9
1700	.0224	15	13	.0211	14	12	.0199	14	10	.0189	13	10	.0179	13	9
1800	.0239	16	14	.0225	15	12	.0213	14	11	.0202	13	10	.0192	12	10
1900	.0255	16	15	.0240	16	13	.0227	15	12	.0215	14	11	.0204	13	10
2000	.0271	17	15	.0256	16	14	.0242	15	13	.0229	14	12	.0217	13	11
2100	.0288	17	16	.0272	16	15	.0257	15	14	.0243	15	13	.0230	14	11
2200	.0305	17	17	.0288	16	16	.0272	15	14	.0258	14	14	.0244	14	12
2300	.0322	18	18	.0304	17	17	.0287	16	15	.0272	15	14	.0258	14	13
2400	.0340	18	19	.0321	17	18	.0303	16	16	.0287	15	15	.0272	14	14
2500	.0358	18	20	.0338	17	19	.0319	16	17	.0302	15	16	.0286	15	14
2600	.0376	19	21	.0355	18	20	.0335	17	18	.0317	16	16	.0301	15	15
2700	.0395	19	22	.0373	18	21	.0352	17	19	.0333	16	17	.0316	15	16
2800	.0414	20	23	.0391	18	22	.0369	17	20	.0349	17	18	.0331	15	16
2900	.0434	20	25	.0409	18	23	.0386	18	20	.0366	17	19	.0346	16	17
3000	.0454	20	27	.0427	19	23	.0404	18	21	.0383	17	20	.0362	16	17
3100	.0474	20	28	.0446	20	24	.0422	18	22	.0400	17	22	.0378	17	18
3200	.0494	21	28	.0466	20	26	.0440	19	23	.0417	18	22	.0395	17	20
3300	.0515	21	29	.0486	20	27	.0459	19	24	.0435	18	23	.0412	17	21
3400	.0536	21	30	.0506	20	28	.0478	19	25	.0453	18	24	.0429	17	22
3500	.0557	22	31	.0526	21	29	.0497	20	26	.0471	18	25	.0446	18	22
3600	.0579	22	32	.0547	21	30	.0517	20	28	.0489	19	25	.0464	19	23

TABLE I.—CONTINUED.—*Auxiliary A.*

z	1950	Δ_z	Δ_r	2000	Δ_z	Δ_r	2050	Δ_z	Δ_r	2100	Δ_z	Δ_r	2150	Δ_z	Δ_r
400	.0035	10	1	.0034	9	2	.0032	8	2	.0030	8	1	.0029	7	1
500	.0045	9	2	.0043	8	3	.0040	8	2	.0038	8	2	.0036	8	1
600	.0054	9	3	.0051	9	3	.0048	9	2	.0046	8	2	.0044	7	2
700	.0063	10	3	.0060	10	3	.0057	9	3	.0054	9	3	.0051	8	2
800	.0073	10	3	.0070	9	4	.0066	9	3	.0063	8	4	.0059	9	2
900	.0083	10	4	.0079	10	4	.0075	9	4	.0071	9	3	.0068	8	3
1000	.0093	11	4	.0089	10	5	.0084	10	4	.0080	9	4	.0076	9	3
1100	.0104	10	5	.0099	10	5	.0094	9	5	.0089	9	4	.0085	9	4
1200	.0114	11	5	.0109	10	6	.0103	10	5	.0098	9	4	.0094	9	4
1300	.0125	11	6	.0119	10	6	.0113	10	6	.0107	10	4	.0103	9	5
1400	.0136	11	7	.0129	11	6	.0123	10	6	.0117	10	5	.0112	9	5
1500	.0147	11	7	.0140	11	7	.0133	11	6	.0127	10	6	.0121	10	5
1600	.0158	12	7	.0151	11	7	.0144	10	7	.0137	10	6	.0131	9	6
1700	.0170	12	8	.0162	11	8	.0154	11	7	.0147	10	7	.0140	10	6
1800	.0182	12	9	.0173	11	8	.0165	10	8	.0157	10	7	.0150	10	7
1900	.0194	12	10	.0184	12	9	.0175	11	8	.0167	11	7	.0160	10	8
2000	.0206	13	10	.0196	12	10	.0186	12	8	.0178	11	8	.0170	10	8
2100	.0219	13	11	.0208	12	10	.0198	12	9	.0189	11	9	.0180	11	8
2200	.0232	13	12	.0220	13	10	.0210	12	10	.0200	11	9	.0191	10	9
2300	.0245	13	12	.0233	13	11	.0222	12	11	.0211	12	10	.0201	11	9
2400	.0258	14	13	.0246	12	12	.0234	12	11	.0223	11	11	.0212	11	9
2500	.0272	14	14	.0258	13	12	.0246	12	12	.0234	12	11	.0223	12	10
2600	.0286	14	15	.0271	14	13	.0258	13	12	.0246	12	11	.0235	11	11
2700	.0300	15	15	.0285	14	14	.0271	13	13	.0258	13	12	.0246	12	11
2800	.0315	15	16	.0299	14	15	.0284	14	13	.0271	13	13	.0258	13	11
2900	.0330	15	17	.0313	14	15	.0298	14	14	.0284	13	13	.0271	12	13
3000	.0345	15	18	.0327	14	15	.0312	14	15	.0297	13	14	.0283	13	13
3100	.0360	15	19	.0341	15	15	.0326	14	16	.0310	14	14	.0296	13	14
3200	.0375	16	19	.0356	16	16	.0340	15	16	.0324	14	15	.0309	13	14
3300	.0391	16	19	.0372	16	17	.0355	15	17	.0338	14	16	.0322	14	15
3400	.0407	17	19	.0388	15	18	.0370	15	18	.0352	14	16	.0336	13	16
3500	.0424	17	21	.0403	16	18	.0385	15	19	.0366	15	17	.0349	14	16
3600	.0441	18	22	.0419	17	19	.0400	16	19	.0381	15	18	.0363	15	16

TABLE I.—Continued.—*Auxiliary A*

z	1200	Δ_z	Δ_v	1250	Δ_z	Δ_v	1300	Δ_z	Δ_v	1350	Δ_z	Δ_v	1400	Δ_z	Δ_v
3600	.1049	36	60	.0989	34	57	.0932	34	54	.0878	31	52	.0826	30	48
3700	.1085	36	62	1023	34	59	.0964	33	55	.0909	31	53	.0856	30	50
3800	.1121	36	64	.1057	35	60	.0997	33	57	.0940	31	54	.0886	30	51
3900	.1157	37	65	.1092	35	62	.1030	34	59	.0971	33	55	.0916	31	52
4000	.1194	38	67	.1127	35	63	.1064	34	60	.1004	33	57	.0947	32	53
4100	.1232	38	70	.1162	36	64	.1098	34	61	.1037	33	58	.0979	32	55
4200	.1270	37	72	.1198	36	66	.1132	34	62	.1070	33	59	.1011	32	57
4300	.1307	38	73	.1234	36	68	.1166	35	63	.1103	34	60	.1043	32	58
4400	.1345	39	75	.1270	37	69	.1201	35	64	.1137	34	62	.1075	33	59
4500	.1384	39	77	.1307	37	71	.1236	36	65	.1171	34	63	.1108	33	61
4600	.1423	39	79	.1344	38	72	.1272	36	67	.1205	35	64	.1141	33	62
4700	.1462	39	80	.1382	38	74	.1308	37	68	.1240	35	66	.1174	34	63
4800	.1501	40	81	.1420	38	75	.1345	37	70	.1275	35	67	.1208	34	65
4900	.1541	40	83	.1458	39	76	.1382	37	72	.1310	36	68	.1242	34	66
5000	.1581	41	84	.1497	39	78	.1419	37	73	.1346	36	70	.1276	34	67
5100	.1622	42	86	.1536	40	80	.1456	38	74	.1382	36	72	.1310	35	68
5200	.1664	41	88	.1576	40	82	.1494	38	76	.1418	37	73	.1345	36	69
5300	.1705	42	89	.1616	40	84	.1532	39	77	.1455	37	74	.1381	36	71
5400	.1747	42	91	.1656	40	85	.1571	39	79	.1492	38	75	.1417	36	72
5500	.1789	43	93	.1696	41	86	.1610	39	80	.1530	38	77	.1453	37	73
5600	.1832	43	95	.1737	41	88	.1649	40	81	.1568	38	78	.1490	37	75
5700	.1875	44	97	.1778	42	89	.1689	40	83	.1606	39	79	.1527	37	77
5800	.1919	44	99	.1820	42	91	.1729	41	84	.1645	39	81	.1564	38	78
5900	.1963	45	101	.1862	43	92	.1770	41	86	.1684	40	82	.1602	38	80
6000	.2008	44	103	.1905	43	94	.1811	41	87	.1724	40	84	.1640	38	81
6100	.2052	44	104	.1948	43	96	.1852	42	88	.1764	40	86	.1678	39	82
6200	.2096	45	105	.1991	43	97	.1894	42	90	.1804	40	87	.1717	39	84
6300	.2141	46	107	.2034	44	98	.1936	43	92	.1844	41	88	.1756	39	85
6400	.2187	47	109	.2078	44	99	.1979	42	94	.1885	41	90	.1795	40	86
6500	.2234	47	112	.2122	45	101	.2021	42	95	.1926	41	91	.1835	40	88
6600	.2281	46	114	.2167	45	104	.2063	44	96	.1967	42	92	.1875	40	89
6700	.2327	47	115	.2212	46	105	.2107	45	98	.2009	43	94	.1915	41	90
6800	.2374	48	116	.2258	47	106	.2152	45	100	.2052	43	96	.1956	42	91

TABLE 1.—CONTINUED.—*Auxiliary A.*

z	1450	Δ_z	Δ_v	1500	Δ_z	Δ_v	1550	Δ_z	Δ_v	1600	Δ_z	Δ_v	1650	Δ_z	Δ_v
3600	.0778	28	46	.0732	27	43	.0689	26	40	.0649	25	36	.0613	23	34
3700	.0806	29	47	.0759	28	44	.0715	27	41	.0674	25	38	.0636	24	35
3800	.0835	29	48	.0787	28	45	.0742	27	43	.0699	26	39	.0660	24	37
3900	.0864	30	49	.0815	28	46	.0769	27	44	.0725	26	41	.0684	25	38
4000	.0894	30	51	.0843	29	47	.0796	27	45	.0751	26	42	.0709	25	39
4100	.0924	30	52	.0872	29	49	.0823	28	46	.0777	27	43	.0734	26	40
4200	.0954	31	53	.0901	29	50	.0851	28	47	.0804	27	44	.0760	26	41
4300	.0985	31	54	.0930	30	51	.0879	28	48	.0831	27	45	.0786	26	42
4400	.1016	31	56	.0960	30	53	.0907	29	49	.0858	28	46	.0812	26	43
4500	.1047	32	57	.0990	30	54	.0936	29	50	.0886	28	48	.0838	27	44
4600	.1079	32	59	.1020	31	55	.0965	30	51	.0914	28	49	.0865	27	45
4700	.1111	32	60	.1051	31	56	.0995	30	52	.0942	29	50	.0892	28	46
4800	.1143	33	61	.1082	31	57	.1025	30	53	.0971	29	51	.0920	28	48
4900	.1176	33	63	.1113	32	58	.1055	31	55	.1000	29	52	.0948	28	49
5000	.1209	33	64	.1145	32	59	.1086	31	57	.1029	30	53	.0976	28	50
5100	.1242	34	65	.1177	33	60	.1117	31	58	.1059	30	55	.1004	29	51
5200	.1276	34	66	.1210	33	62	.1148	31	59	.1089	30	56	.1033	29	52
5300	.1310	35	67	.1243	33	64	.1179	32	60	.1119	31	57	.1062	30	53
5400	.1345	35	69	.1276	34	65	.1211	32	61	.1150	31	58	.1092	30	55
5500	.1380	35	70	.1310	34	67	.1243	33	62	.1181	32	59	.1122	30	56
5600	.1415	35	71	.1344	34	68	.1276	33	63	.1213	32	61	.1152	31	57
5700	.1450	36	72	.1378	34	69	.1309	34	64	.1245	32	62	.1183	31	58
5800	.1486	36	74	.1412	35	69	.1343	34	66	.1277	32	63	.1214	31	59
5900	.1522	37	75	.1447	35	70	.1377	34	68	.1309	33	64	.1245	32	60
6000	.1559	37	77	.1482	36	71	.1411	34	69	.1342	33	65	.1277	32	61
6100	.1596	37	78	.1518	36	73	.1445	35	70	.1375	34	66	.1309	33	62
6200	.1633	38	79	.1554	37	74	.1480	35	71	.1409	34	67	.1342	33	64
6300	.1671	38	80	.1591	37	76	.1515	36	72	.1443	34	68	.1375	33	65
6400	.1709	38	81	.1628	37	77	.1551	36	74	.1477	35	69	.1408	33	66
6500	.1747	39	82	.1665	37	78	.1587	36	75	.1512	35	71	.1441	34	66
6600	.1786	39	84	.1702	38	79	.1623	36	76	.1547	36	72	.1475	34	67
6700	.1825	40	85	.1740	39	81	.1659	37	76	.1583	36	74	.1509	35	69
6800	.1865	40	86	.1779	38	83	.1696	37	77	.1619	35	75	.1544	35	70

TABLE I.—CONTINUED.—*Auxiliary A.*

z.	1700	Δ_v	Δ_z	1750	Δ_z	Δ_v	1800	Δ_z	Δ_v	1850	Δ_z	Δ_v	1900	Δ_z	Δ_v
3600	.0579	22	32	.0547	21	30	.0517	20	28	.0489	19	25	.0464	18	23
3700	.0601	22	33	.0568	22	31	.0537	21	29	.0508	20	26	.0482	19	25
3800	.0623	23	33	.0590	22	32	.0558	21	30	.0528	20	27	.0501	19	25
3900	.0646	24	34	.0612	22	33	.0579	21	31	.0548	20	28	.0520	20	26
4000	.0670	24	36	.0634	23	34	.0600	22	32	.0568	21	28	.0540	19	27
4100	.0694	25	37	.0657	23	35	.0622	22	33	.0589	21	30	.0559	20	27
4200	.0719	25	39	.0680	23	36	.0644	22	34	.0610	22	31	.0579	21	28
4300	.0744	25	41	.0703	24	37	.0666	23	34	.0632	22	32	.0600	21	30
4400	.0769	25	42	.0727	24	38	.0689	22	35	.0654	22	33	.0621	21	31
4500	.0794	26	43	.0751	25	40	.0711	23	35	.0676	22	34	.0642	21	32
4600	.0820	26	44	.0776	25	42	.0734	25	36	.0698	23	35	.0663	22	32
4700	.0846	26	45	.0801	25	42	.0759	25	38	.0721	23	36	.0685	22	33
4800	.0872	27	46	.0826	26	42	.0784	25	40	.0744	23	37	.0707	23	34
4900	.0899	27	47	.0852	26	43	.0809	25	42	.0767	24	37	.0730	23	35
5000	.0926	27	48	.0878	26	44	.0834	25	43	.0791	25	38	.0753	23	36
5100	.0953	28	49	.0964	27	45	.0859	25	43	.0816	25	40	.0776	24	37
5200	.0981	28	40	.0931	27	47	.0884	26	43	.0841	25	41	.0800	24	39
5300	.1009	28	51	.0958	28	48	.0910	27	44	.0866	25	42	.0824	25	40
5400	.1037	29	51	.0986	28	49	.0937	27	46	.0891	26	42	.0849	25	41
5500	.1066	29	52	.1014	28	50	.0964	27	47	.0917	26	43	.0874	25	42
5600	.1095	30	53	.1042	28	51	.0991	27	48	.0943	27	44	.0899	25	43
5700	.1125	30	55	.1070	29	52	.1018	28	48	.0970	27	46	.0924	26	43
5800	.1155	30	56	.1099	29	53	.1046	28	49	.0997	27	47	.0950	26	44
5900	.1185	31	57	.1128	30	54	.1074	29	50	.1024	27	48	.0976	27	45
6000	.1216	31	58	.1158	30	55	.1103	29	52	.1051	28	48	.1003	26	46
6100	.1247	31	59	.1188	30	56	.1132	29	53	.1079	28	50	.1029	27	47
6200	.1278	32	60	.1218	31	57	.1161	30	54	.1107	29	51	.1056	28	48
6300	.1310	32	61	.1249	31	58	.1191	30	55	.1136	29	52	.1084	28	49
6400	.1342	33	62	.1280	31	59	.1221	30	56	.1165	29	53	.1112	28	50
6500	.1375	33	64	.1311	31	60	.1251	31	57	.1194	30	54	.1140	29	50
6600	.1408	32	66	.1342	32	60	.1282	31	58	.1224	30	55	.1169	29	51
6700	.1440	33	66	.1374	33	61	.1313	31	59	.1254	30	56	.1198	29	52
6800	.1473	34	66	.1407	32	63	.1344	31	60	.1284	31	57	.1227	30	53

TABLE I.—Continued.—*Auxiliary A.*

z	1950	Δ_z	Δ_v	2000	Δ_z	Δ_v	2050	Δ_z	Δ_v	2100	Δ_z	Δ_v	2150	Δ_z	Δ_v
3600	.0441	18	22	.0419	17	20	.0399	16	18	.0381	15	18	.0363	15	16
3700	.0459	17	23	.0436	17	21	.0415	16	19	.0396	15	18	.0378	14	18
3800	.0476	18	23	.0453	17	22	.0431	16	20	.0411	15	19	.0392	15	18
3900	.0494	19	24	.0470	18	23	.0447	17	21	.0426	16	19	.0407	15	19
4000	.0513	19	25	.0488	18	24	.0464	17	22	.0442	16	20	.0422	16	19
4100	.0532	19	26	.0506	18	25	.0481	18	23	.0458	17	20	.0438	16	20
4200	.0551	19	27	.0524	18	26	.0499	17	24	.0475	17	21	.0454	16	21
4300	.0570	20	28	.0542	19	26	.0516	18	24	.0492	17	22	.0470	16	22
4400	.0590	20	28	.0561	19	27	.0534	19	25	.0509	18	23	.0486	17	22
4500	.0610	21	30	.0580	20	27	.0553	19	26	.0527	18	24	.0503	17	23
4600	.0631	21	31	.0600	20	28	.0572	19	27	.0545	18	25	.0520	18	24
4701	.0652	21	32	.0620	20	29	.0591	19	28	.0563	19	25	.0538	18	25
4800	.0673	22	33	.0640	21	30	.0610	20	28	0582	19	26	.0556	18	26
4900	.0695	22	34	.0661	21	31	.0630	20	29	.0601	19	27	.0574	19	27
5000	.0717	22	35	.0682	21	32	.0650	21	30	.0620	20	27	.0593	19	28
5100	.0739	22	36	.0703	22	32	.0671	21	31	.0640	20	28	.0612	19	29
5200	.0761	23	36	.0725	22	33	.0692	21	32	.0660	20	29	.0631	19	29
5300	.0784	24	37	.0747	23	34	.0713	22	33	.0680	20	30	.0650	20	29
5400	.0808	24	38	.0770	23	35	.0735	22	34	.0701	21	31	.0670	20	30
5500	.0832	24	39	.0793	23	36	.0757	22	35	.0722	21	32	.0690	21	30
5600	.0856	25	40	.0816	24	37	.0779	23	35	.0744	22	33	.0711	22	31
5700	.0881	25	41	.0840	24	38	.0802	23	36	.0766	22	33	.0733	21	33
5800	.0906	25	42	.0864	24	39	.0825	24	37	.0788	22	34	.0754	21	33
5900	.0931	26	43	.0888	25	39	.0849	24	39	.0810	23	35	.0775	22	33
6000	.0957	25	44	.0913	25	40	.0873	24	40	.0833	24	36	.0797	23	34
6100	.0982	26	45	.0938	25	41	.0897	24	40	.0857	24	37	.0820	23	36
6200	.1008	27	45	.0963	26	42	.0921	25	40	.0881	24	38	.0843	23	37
6300	.1035	27	46	.0989	26	43	.0946	25	41	.0905	24	39	.0866	23	38
6400	.1062	28	47	.1015	26	44	.0971	25	42	.0929	24	40	.0889	23	38
6500	.1090	28	49	.1041	27	45	.0996	26	43	.0953	25	41	.0912	24	38
6600	.1118	28	50	.1068	27	47	.1022	26	44	.0978	25	42	.0936	25	38
6700	.1146	28	51	.1095	28	47	.1048	27	45	.1003	26	42	.0961	26	40
6800	.1174	29	51	.1123	28	48	.1075	27	46	.1029	26	42	.0987	25	42

TABLE I.—Continued.—*Auxiliary B.*

z	1200	Δ_z	Δ_v	1250	Δ_z	Δ_v	1300	Δ_z	Δ_v	1350	Δ_z	Δ_v	1400	Δ_z	Δ_v
400	.0096	26	7	.0089	23	7	.0082	22	6	.0076	21	5	.0071	20	5
500	.0122	27	10	.0112	25	8	.0104	23	7	.0097	22	6	.0091	20	7
600	.0149	27	12	.0137	26	10	.0127	24	8	.0119	22	8	.0111	21	8
700	.0176	28	13	.0163	26	12	.0151	25	10	.0141	23	9	.0132	21	10
800	.0204	29	15	.0189	27	13	.0176	26	12	.0164	24	11	.0153	22	10
900	.0233	30	17	.0216	28	14	.0202	26	14	.0188	25	13	.0175	24	12
1000	.0263	31	19	.0244	30	16	.0228	28	15	.0213	26	14	.0199	24	13
1100	.0294	31	20	.0274	30	18	.0256	29	17	.0239	27	16	.0223	24	15
1200	.0325	32	21	.0304	30	19	.0285	28	19	.0266	27	19	.0247	26	16
1300	.0357	33	23	.0334	31	21	.0313	29	20	.0293	27	20	.0273	26	17
1400	.0390	34	25	.0365	32	23	.0342	31	22	.0320	29	21	.0299	27	18
1500	.0424	34	27	.0397	33	24	.0373	31	24	.0349	30	23	.0326	28	20
1600	.0458	35	28	.0430	33	26	.0404	32	25	.0379	30	25	.0354	29	21
1700	.0493	36	30	.0463	34	27	.0436	32	27	.0409	31	26	.0383	30	23
1800	.0529	36	32	.0497	35	29	.0468	33	28	.0440	32	27	.0413	30	25
1900	.0565	36	33	.0532	35	31	.0501	34	29	.0472	32	29	.0443	31	26
2000	.0601	37	34	.0567	36	32	.0535	35	31	.0504	34	30	.0474	33	28
2100	.0638	38	35	.0603	37	33	.9570	35	32	.0538	34	31	.0507	33	30
2200	.0676	39	36	.0640	37	35	.0605	36	33	.0572	35	32	.0540	33	31
2300	.0715	39	38	.0677	38	36	.0641	37	34	.0607	35	34	.0573	34	32
2400	.0754	40	39	0715	38	37	.0678	37	36	.0642	36	35	.0607	35	34
2500	.0794	40	41	.0753	39	38	.0715	38	37	.0678	37	36	.0642	35	35
2600	.0834	41	42	.0792	40	39	.0753	38	38	.0715	37	38	.0677	36	36
2700	.0875	42	43	.0832	40	41	.0791	39	39	.0752	38	39	.0713	37	37
2800	.0917	43	45	.0872	41	42	.0830	40	40	.0790	39	40	.0750	38	39
2900	.0960	43	47	.0913	42	43	.0870	40	41	.0829	39	41	.0788	38	40
3000	.1003	44	48	.0955	43	45	.0910	41	42	.0868	40	42	.0826	39	41
3100	.1047	45	49	.0998	43	47	.0951	42	43	.0908	40	43	.0865	39	42
3200	.1092	46	51	.1041	44	48	.0993	42	45	.0948	41	44	.0904	40	43
3300	.1138	46	53	.1085	44	50	.1035	43	46	.0989	42	45	.0944	40	44
3400	.1184	47	55	.1129	46	51	.1078	44	47	.1031	42	47	.0984	41	45
3500	.1231	48	56	.1175	47	53	.1122	45	49	.1073	43	48	.1025	42	46
3600	.1279	49	57	.1222	47	55	.1167	45	51	.1116	44	49	.1067	43	47

TABLE I.—CONTINUED.—*Auxiliary B.*

z	1450	Δ_z	Δ_v	1500	Δ_z	Δ_v	1550	Δ_z	Δ_v	1600	Δ_z	Δ_v	1650	Δ_z	Δ_v
400	.0066	18	5	.0061	17	4	.0057	16	3	.0054	15	3	.0051	14	3
500	.0084	19	6	.0078	18	5	.0073	17	4	.0069	15	4	.0065	14	4
600	.0103	19	7	.0096	18	6	.0090	17	6	.0084	16	5	.0079	15	4
700	.0122	19	8	.0114	19	7	.0107	18	7	.0100	17	6	.0094	16	5
800	.0141	22	10	.0133	20	8	.0125	18	8	.0117	17	7	.0110	16	7
900	.0163	23	12	.0153	20	10	.0143	19	9	.0134	18	8	.0126	17	7
1000	.0186	22	13	.0173	21	11	.0162	20	10	.0152	19	9	.0143	17	8
1100	.0208	23	14	.0194	22	12	.0182	20	11	.0171	19	11	.0160	18	9
1200	.0231	25	15	.0216	23	14	.0202	22	12	.0190	20	12	.0178	19	10
1300	.0256	25	17	.0239	24	15	.0224	22	14	.0210	20	13	.0197	19	11
1400	.0281	25	18	.0263	24	17	.0246	23	16	.0230	22	14	.0216	20	12
1500	.0306	27	19	.0287	25	18	.0269	24	17	.0252	22	16	.0236	21	13
1600	.0333	27	21	.0312	26	19	.0293	24	19	.0274	23	17	.0257	22	15
1700	.0360	28	22	.0338	26	21	.0317	24	20	.0297	23	18	.0279	22	16
1800	.0388	29	24	.0364	27	23	.0341	26	21	.0320	25	19	.0301	23	17
1900	.0417	29	26	.0391	28	24	.0367	27	22	.0345	25	21	.0324	24	18
2000	.0446	31	27	.0419	29	25	.0394	27	24	.0370	26	22	.0348	25	20
2100	.0477	32	29	.0448	30	27	.0421	28	25	.0396	27	23	.0373	25	22
2200	.0509	32	31	.0478	31	29	.0449	29	26	.0423	28	25	.0398	27	23
2300	.0541	32	32	.0509	31	31	.0478	30	27	.0451	28	26	.0425	27	25
2400	.0573	34	33	.0540	32	32	.0508	31	29	.0479	29	27	.0452	27	26
2500	.0607	34	35	.0572	33	33	.0539	31	31	.0508	30	29	.0479	28	27
2600	.0641	35	36	.0605	34	35	.0570	33	32	.0538	31	31	.0507	30	28
2700	.0676	35	37	.0639	34	36	.0603	33	34	.0569	32	32	.0537	30	30
2800	.0711	37	38	.0673	35	37	.0636	34	35	.0601	33	34	.0567	31	31
2900	.0748	37	40	.0708	36	38	.0670	35	36	.0634	33	36	.0598	32	32
3000	.0785	38	41	.0744	37	39	.0705	35	38	.0667	34	37	.0630	33	34
3100	.0823	38	42	.0781	37	41	.0740	36	39	.0701	35	38	.0663	34	35
3200	.0861	39	43	.0818	38	42	.0776	37	40	.0736	36	39	.0697	35	37
3300	.0900	39	44	.0856	38	43	.0813	37	41	.0772	36	40	.0732	35	38
3400	.0939	40	45	.0894	39	44	.0850	38	42	.0808	37	41	.0767	36	39
3500	.0979	41	46	.0933	40	45	.0888	39	43	.0845	38	42	.0803	37	40
3600	.1020	41	47	.0973	40	46	.0927	39	44	.0883	38	43	.0840	37	42

TABLE I.—CONTINUED.—*Auxiliary B.*

z	1700	Δ_z	Δ_v	1750	Δ_z	Δ_v	1800	Δ_z	Δ_v	1850	Δ_z	Δ_v	1900	Δ_z	Δ_v
400	.0048	13	3	.0045	13	2	.0043	12	2	.0041	11	2	.0039	10	2
500	.0061	14	3	.0058	13	3	.0055	12	3	.0052	11	3	.0049	11	2
600	.0075	14	4	.0071	13	4	.0067	12	4	.0063	12	3	.0060	11	3
700	.0089	14	5	.0084	13	5	.0079	13	4	.0075	12	4	.0071	12	[illegible]
800	.0103	16	6	.0097	15	5	.0092	14	5	.0087	13	4	.0083	12	4
900	.0119	16	7	.0112	15	6	.0106	14	6	.0100	14	5	.0095	13	5
1000	.0135	16	8	.0127	16	7	.0120	15	6	.0114	14	6	.0108	13	6
1100	.0151	17	8	.0143	16	8	.0135	15	7	.0128	14	7	.0121	14	6
1200	.0168	18	9	.0159	17	9	.0150	16	8	.0142	15	7	.0135	14	7
1300	.0186	18	10	.0176	17	10	.0166	16	9	.0157	15	8	.0149	14	8
1400	.0204	19	11	.0193	17	11	.0182	17	10	.0172	16	9	.0163	15	8
1500	.0223	19	13	.0210	18	11	.0199	17	11	.0188	17	10	.0178	16	9
1600	.0242	21	14	.0228	20	12	.0216	18	11	.0205	17	11	.0194	16	10
1700	.0263	21	15	.0248	20	14	.0234	19	12	.0222	17	12	.0210	17	10
1800	.0284	22	16	.0268	21	15	.0253	20	14	.0239	19	12	.0227	18	11
1900	.0306	22	17	.0289	21	16	.0273	20	15	.0258	19	13	.0245	18	13
2000	.0328	23	18	.0310	21	17	.0293	20	16	.0277	19	14	.0263	18	14
2100	.0351	24	20	.0331	23	18	.0313	22	17	.0296	21	15	.0281	19	14
2200	.0375	25	21	.0354	24	19	.0335	22	18	.0317	21	17	.0300	20	15
2300	.0400	26	22	.0378	24	21	.0357	23	19	.0338	22	18	.0320	21	16
2400	.0426	26	24	.0402	25	22	.0380	24	20	.0360	23	19	.0341	22	17
2500	.0452	27	25	.0427	26	23	.0404	24	21	.0383	23	20	.0363	22	18
2600	.0479	28	26	.0453	26	25	.0428	26	22	.0406	24	21	.0385	23	19
2700	.0507	29	28	.0479	28	25	.0454	26	24	.0430	25	22	.0408	23	21
2800	.0536	30	29	.0507	29	27	.0480	27	25	.0455	25	24	.0431	25	22
2900	.0566	30	30	.0536	29	29	.0507	28	27	.0480	27	25	.0456	25	23
3000	.0596	32	31	.0565	30	30	.0535	29	28	.0507	27	26	.0481	26	24
3100	.0628	32	33	.0595	31	31	.0564	30	30	.0534	29	27	.0507	27	25
3200	.0660	34	34	.0626	32	32	.0594	30	31	.0563	29	29	.0534	27	27
3300	.0694	34	36	.0658	32	34	.0624	31	32	.0592	29	30	.0561	29	28
3400	.0728	35	38	.0690	34	35	.0655	32	34	.0621	31	31	.0590	30	30
3500	.0763	35	39	.0724	34	37	.0687	33	35	.0652	32	32	.0620	30	31
3600	.0798	36	40	.0758	34	38	.0720	34	36	.0684	32	34	.0650	31	32

TABLE I.—CONTINUED.—*Auxiliary B.*

z	1950	Δ_s	Δ_c	2000	Δ_s	Δ_c	2050	Δ_s	Δ_c	2100	Δ_s	Δ_c	2150	Δ_s	Δ_c
400	.0037	10	1	.0036	10	2	.0034	9	2	.0032	8	2	.0030	8	2
500	.0047	10	1	.0046	10	3	.0043	9	3	.0040	9	2	.0038	9	2
600	.0057	11	1	.0056	9	4	.0052	9	3	.0049	9	2	.0047	8	2
700	.0068	11	3	.0065	10	4	.0061	10	3	.0058	10	3	.0055	9	2
800	.0079	11	4	.0075	11	4	.0071	10	3	.0068	10	4	.0064	10	2
900	.0090	12	4	.0086	11	5	.0081	11	3	.0078	10	4	.0074	10	3
1000	.0102	13	5	.0097	12	5	.0092	11	4	.0088	11	4	.0084	10	4
1100	.0115	13	6	.0109	12	5	.0103	12	5	.0099	11	5	.0094	11	4
1200	.0128	13	7	.0121	13	6	.0115	12	5	.0110	12	5	.0105	11	5
1300	.0141	14	7	.0134	13	7	.0127	13	5	.0122	12	6	.0116	11	5
1400	.0155	14	8	.0147	14	7	.0140	13	6	.0134	12	7	.0127	12	5
1500	.0169	15	8	.0161	14	8	.0153	14	7	.0146	13	7	.0139	12	6
1600	.0184	16	9	.0175	15	8	.0167	14	8	.0159	13	8	.0151	13	6
1700	.0200	16	10	.0190	15	9	.0181	14	9	.0172	14	8	.0164	13	7
1800	.0216	16	11	.0205	16	10	.0195	15	9	.0186	14	9	.0177	14	7
1900	.0232	17	11	.0221	16	11	.0210	16	10	.0200	15	9	.0191	14	8
2000	.0249	18	12	.0237	17	11	.0226	16	11	.0215	15	10	.0205	15	9
2100	.0267	18	13	.0254	17	12	.0242	16	12	.0230	16	10	.0220	15	10
2200	.0285	19	14	.0271	18	13	.0258	17	12	.0246	16	11	.0235	15	11
2300	.0304	20	15	.0289	19	14	.0275	18	13	.0262	17	12	.0250	16	11
2400	.0324	21	16	.0308	20	15	.0293	19	14	.0279	18	13	.0266	17	12
2500	.0345	21	17	.0328	20	16	.0312	19	15	.0297	18	14	.0283	18	13
2600	.0366	21	18	.0348	20	17	.0331	19	16	.0315	19	14	.0301	18	14
2700	.0387	22	19	.4368	21	18	.0350	20	16	.0334	19	15	.0319	18	15
2800	.0409	24	20	.0389	22	19	.0370	21	17	.0353	20	16	.0337	19	15
2900	.0433	24	22	.0411	23	20	.0391	22	18	.0373	20	17	.0356	19	16
3000	.0457	25	23	.0434	24	21	.0413	23	20	.0393	22	18	.0375	21	16
3100	.0482	25	24	.0458	24	22	.0436	23	21	.0415	22	19	.0396	21	18
3200	.0507	26	25	.0482	25	23	.0459	24	22	.0437	23	20	.0417	22	19
3300	.0533	27	26	.0507	26	24	.0483	25	23	.0460	24	21	.0439	22	20
3400	.0560	29	27	.0533	27	25	.0508	26	24	.0484	25	23	.0461	24	21
3500	.0589	29	29	.0560	28	26	.0534	26	25	.0509	25	24	.0485	24	22
3600	.0618	30	30	.0588	29	28	.0560	27	26	.0534	26	25	.0509	26	23

TABLE I.—Continued.—*Auxiliary B.*

z	1200	Δ_z	Δ_v	1250	Δ_z	Δ_v	1300	Δ_z	Δ_v	1350	Δ_z	Δ_v	1400	Δ_z	Δ_v
3600	.1279	49	57	.1222	47	55	.1167	45	51	.1116	44	49	.1067	43	47
3700	.1328	49	59	.1269	47	57	.1212	46	52	.1160	45	50	.1110	43	49
3800	.1377	50	61	.1316	48	58	.1258	47	53	.1205	45	52	.1153	44	50
3900	.1427	50	63	.1364	48	59	.1305	47	55	.1250	46	53	.1197	45	51
4000	.1477	52	65	.1412	50	60	.1352	48	56	.1296	47	54	.1242	45	53
4100	.1529	52	67	.1462	50	62	.1400	49	57	.1343	47	56	.1287	46	54
4200	.1581	53	69	.1512	52	63	.1449	50	59	.1390	48	57	.1333	47	55
4300	.1634	54	70	.1564	52	65	.1499	50	61	.1438	49	58	.1380	47	56
4400	.1688	55	72	.1616	52	67	.1549	51	62	.1487	50	60	.1427	48	57
4500	.1743	55	75	.1668	54	68	.1600	52	63	.1537	50	62	.1475	49	59
4600	.1798	57	76	.1722	55	70	.1652	53	65	.1587	51	63	.1524	50	61
4700	.1855	57	78	.1777	55	72	.1705	53	67	.1638	52	64	.1574	51	62
4800	.1912	58	80	.1832	56	74	.1758	54	68	.1690	53	65	.1625	51	64
4900	.1970	59	82	.1888	57	76	.1812	55	69	.1743	53	67	.1676	52	65
5000	.2029	60	84	.1945	58	78	.1867	56	71	.1796	55	68	.1728	53	67
5100	.2089	60	86	.2003	58	80	.1923	57	72	.1851	55	70	.1781	53	68
5200	.2149	61	88	.2061	59	81	.1980	58	74	.1906	56	72	.1834	54	69
5300	.2210	62	90	.2120	60	82	.2038	58	76	.1962	56	74	.1888	55	70
5400	.2272	63	92	.2180	61	84	.2096	59	78	.2018	58	75	.1943	56	72
5500	.2335	64	94	.2241	62	86	.2155	60	79	.2076	58	77	.1999	57	73
5600	.2399	64	96	.2303	63	88	.2215	61	81	.2134	59	78	.2056	57	75
5700	.2403	66	99	.2366	63	90	.2276	61	83	.2193	59	80	.2113	58	76
5800	.2529	67	100	.2429	65	92	.2337	63	85	.2252	61	81	.2171	59	78
5900	.2596	68	102	.2494	65	94	.2400	63	87	.2313	62	83	.2230	60	79
6000	.2664	68	105	.2559	66	96	.2463	64	88	.2375	62	85	.2290	61	81
6100	.2732	69	107	.2625	67	98	.2527	65	90	.2437	63	86	.2351	61	83
6200	.2801	71	109	.2692	68	100	.2592	66	92	.2500	65	88	.2412	63	84
6300	.2872	71	112	.2760	69	102	.2658	67	93	.2565	65	90	.2475	63	86
6400	.2943	73	114	.2829	69	104	.2725	67	95	.2630	66	92	.2538	64	87
6500	.3016	73	118	.2898	71	106	.2792	69	96	.2696	66	94	.2602	65	89
6600	.3089	74	120	.2969	72	108	.2861	70	99	.2762	68	95	.2667	66	91
6700	.3163	75	122	.3041	73	110	.2931	71	101	.2830	68	97	.2733	66	93
6800	.3238	77	124	.3114	74	112	.3002	72	104	.2898	70	99	.2799	68	94

TABLE I.—CONTINUED.—*Auxiliary B.*

z	1450	Δ_z	Δ_v	1500	Δ_z	Δ_v	1550	Δ_z	Δ_v	1600	Δ_z	Δ_v	1650	Δ_z	Δ_v
3600	.1020	41	47	.0973	40	46	.0927	39	44	.0883	38	43	.0840	37	42
3700	.1061	42	48	.1013	41	47	.0966	40	45	.0921	39	44	.0877	38	43
3800	.1103	43	49	.1054	42	48	.1006	41	46	.0960	40	45	.0915	39	44
3900	.1146	43	50	.1096	42	49	.1047	41	47	.1000	40	46	.0954	39	45
4000	.1189	44	51	.1138	43	50	.1088	42	48	.1040	41	47	.0993	40	46
4100	.1233	45	52	.1181	44	51	.1130	42	49	.1081	41	48	.1033	40	47
4200	.1278	46	53	.1225	44	52	.1172	43	50	.1122	42	49	.1073	41	48
4300	.1324	46	55	.1269	45	54	.1215	44	51	.1164	43	50	.1114	42	49
4400	.1370	46	56	.1314	45	55	.1259	45	52	.1207	44	51	.1156	42	50
4500	.1416	47	57	.1359	46	55	.1304	46	53	.1251	44	53	.1198	43	51
4600	.1463	49	58	.1405	47	55	.1350	45	55	.1295	45	54	.1241	44	52
4700	.1512	49	60	.1452	48	57	.1395	46	55	.1340	45	55	.1285	44	53
4800	.1561	50	61	.1500	48	59	.1441	47	56	.1385	46	56	.1329	45	54
4900	.1611	50	63	.1548	49	60	.1488	48	57	.1431	47	57	.1374	46	55
5000	.1661	52	64	.1597	50	61	.1536	49	58	.1478	47	58	.1420	46	56
5100	.1713	52	66	.1647	51	62	.1585	50	60	.1525	48	59	.1466	47	57
5200	.1765	53	67	.1698	52	63	.1635	49	62	.1573	48	60	.1513	48	58
5300	.1818	53	68	.1750	52	66	.1684	51	63	.1621	50	60	.1561	48	59
5400	.1871	55	69	.1802	53	67	.1735	52	64	.1671	51	62	.1609	49	60
5500	.1926	55	71	.1855	54	68	.1787	52	65	.1722	51	64	.1658	50	61
5600	.1981	56	72	.1909	54	70	.1839	52	66	.1773	52	65	.1708	51	62
5700	.2037	56	74	.1963	55	72	.1891	54	66	.1825	52	66	.1759	51	63
5800	.2093	58	75	.2018	56	73	.1945	55	68	.1877	53	67	.1810	52	64
5900	.2151	58	77	.2074	57	74	.2000	56	70	.1930	54	68	.1862	53	65
6000	.2209	59	78	.2131	58	75	.2056	57	72	.1984	55	69	.1915	54	67
6100	.2268	60	79	.2189	58	76	.2113	57	74	.2039	56	70	.1969	54	68
6200	.2328	61	81	.2247	60	77	.2170	58	75	.2095	57	72	.2023	55	69
6300	.2389	62	82	.2307	60	79	.2228	58	76	.2152	57	74	.2078	56	70
6400	.2451	62	84	.2367	61	81	.2286	60	77	.2209	58	75	.2134	57	72
6500	.2513	63	85	.2428	61	82	.2346	60	79	.2267	59	76	.2191	57	73
6600	.2576	64	87	.2489	63	83	.2406	60	80	.2326	60	78	.2248	59	74
6700	.2640	65	88	.2552	63	86	.2466	62	80	.2386	60	79	.2307	59	76
6800	.2705	66	90	.2615	64	87	.2528	63	82	.2446	61	80	.2366	60	77

TABLE I.—Continued.—*Auxiliary B.*

z	1700	Δ_z	Δ_v	1750	Δ_z	Δ_v	1800	Δ_z	Δ_v	1850	Δ_z	Δ_v	1900	Δ_z	Δ_v
3600	.0798	36	40	.0758	34	38	.0720	34	36	.0684	32	34	.0650	31	32
3700	.0834	37	42	.0792	36	38	.0754	34	38	.0716	33	36	.0681	32	33
3800	.0871	38	43	.0828	37	40	.0788	36	39	.0749	34	36	.0713	33	35
3900	.0909	38	44	.0865	38	41	.0824	36	41	.0783	35	37	.0746	33	36
4000	.0947	39	44	.0903	38	43	.0860	37	42	.0818	36	39	.0779	35	37
4100	.0986	39	45	.0941	38	44	.0897	37	43	.0854	37	40	.0814	35	38
4200	.1025	40	46	.0979	39	45	.0934	38	43	.0891	37	42	.0849	36	39
4300	.1065	41	47	.1018	40	46	.0972	39	44	.0928	38	43	.0885	37	40
4400	.1106	41	48	.1058	41	47	.1011	40	45	.0966	39	44	.0922	38	42
4500	.1147	42	48	.1099	41	48	.1051	40	46	.1005	39	45	.0960	38	43
4600	.1189	43	49	.1140	42	49	.1091	41	47	.1044	40	46	.0998	39	44
4700	.1232	43	50	.1182	42	50	.1132	41	48	.1084	40	47	.1037	40	45
4800	.1275	44	51	.1224	43	51	.1173	42	49	.1124	41	47	.1077	40	47
4900	.1319	45	52	.1267	43	52	.1215	43	50	.1165	42	48	.1117	41	47
5000	.1364	45	54	.1310	45	52	.1258	43	51	.1207	43	49	.1158	42	48
5100	.1409	46	54	.1355	45	54	.1301	44	51	.1250	43	50	.1200	42	49
5200	.1455	47	55	.1400	45	55	.1345	45	52	.1293	44	51	.1242	43	49
5300	.1502	47	57	.1445	46	55	.1390	45	53	.1337	44	52	.1285	43	50
5400	.1549	48	58	.1491	47	56	.1435	46	54	.1381	45	53	.1328	44	51
5500	.1597	49	59	.1538	48	57	.1481	47	55	.1426	45	54	.1372	45	52
5600	.1646	50	60	.1586	48	58	.1528	47	57	.1471	47	54	.1417	45	53
5700	.1696	50	62	.1634	49	59	.1575	48	57	.1518	47	56	.1462	46	54
5800	.1746	51	63	.1683	50	60	.1623	49	58	.1565	48	57	.1508	47	55
5900	.1797	51	64	.1733	51	61	.1672	49	59	.1613	48	58	.1555	48	56
6000	.1848	53	64	.1784	51	63	.1721	50	60	.1661	49	58	.1603	48	57
6100	.1901	53	66	.1835	52	64	.1771	51	61	.1710	49	59	.1651	48	58
6200	.1954	54	67	.1887	53	65	.1822	52	63	.1759	51	60	.1699	50	58
6300	.2008	54	68	.1940	53	66	.1874	52	64	.1810	51	61	.1749	50	60
6400	.2062	56	69	.1993	54	67	.1926	53	65	.1861	52	62	.1799	51	61
6500	.2118	56	71	.2047	55	68	.1979	54	66	.1913	53	63	.1850	52	62
6600	.2174	57	72	.2102	56	69	.2033	55	67	.1966	54	64	.1902	52	64
6700	.2231	58	73	.2158	56	70	.2088	55	68	.2020	54	66	.1954	53	65
6800	.2289	59	75	.2214	57	71	.2143	56	69	.2074	55	67	.2007	54	65

TABLE I.—CONTINUED.—*Auxiliary B.*

z	1950	Δ_z	Δ_v	2000	Δ_z	Δ_v	2050	Δ_z	Δ_v	2100	Δ_z	Δ_v	2150	Δ_z	Δ_v
3600	.0618	30	30	.0588	29	28	.0560	27	26	.0534	26	25	.0509	25	23
3700	.0648	30	31	.0617	29	30	.0587	29	27	.0560	27	27	.0534	26	24
3800	.0678	32	32	.0646	30	30	.0616	29	29	.0587	27	27	.0560	26	25
3900	.0710	32	34	.0676	31	31	.0645	29	31	.0614	29	28	.0586	28	25
4000	.0742	34	35	.0707	32	33	.0674	31	31	.0643	30	29	.0614	28	27
4100	.0776	34	37	.0739	33	34	.0705	31	32	.0673	30	31	.0642	29	28
4200	.0810	35	38	.0772	34	36	.0736	33	33	.0703	31	32	.0671	30	29
4300	.0845	35	39	.0806	34	37	.0769	33	35	.0734	32	33	.0701	31	31
4400	.0880	37	40	.0840	36	38	.0802	34	36	.0766	33	34	.0732	31	32
4500	.0917	37	41	.0876	36	40	.0836	35	37	.0799	34	36	.0763	33	32
4600	.0954	38	42	.0912	37	41	.0871	36	38	.0833	34	37	.0796	33	34
4700	.0992	38	43	.0949	37	42	.0907	37	40	.0867	36	38	.0829	34	35
4800	.1030	40	44	.0986	39	42	.0944	37	41	.0903	37	40	.0863	36	36
4900	.1070	40	45	.1025	39	44	.0981	38	41	.0940	37	41	.0899	36	38
5000	.1110	41	46	.1064	40	45	.1019	39	42	.0977	38	42	.0935	37	39
5100	.1151	42	47	.1104	40	46	.1058	40	43	.1015	38	43	.0972	38	41
5200	.1193	42	49	.1144	41	46	.1098	40	45	.1053	39	43	.1010	38	42
5300	.1235	42	50	.1185	42	47	.1138	41	46	.1092	40	44	.1048	39	42
5400	.1277	43	50	.1227	42	48	.1179	41	47	.1132	41	45	.1087	40	43
5500	.1320	44	51	.1269	43	49	.1220	42	47	.1173	41	46	.1127	40	44
5600	.1364	44	52	.1312	44	50	.1262	43	48	.1214	42	47	.1167	40	45
5700	.1408	45	52	.1356	44	51	.1305	44	49	.1256	43	49	.1207	42	45
5800	.1453	46	53	.1400	45	51	.1349	44	50	.1299	43	50	.1249	43	46
5900	.1499	47	54	.1445	46	52	.1393	44	51	.1342	44	49	.1292	43	47
6000	.1546	47	55	.1491	46	54	.1437	45	51	.1386	45	50	.1335	44	48
6100	.1593	48	56	.1537	47	55	.1482	46	51	.1431	45	52	.1379	44	49
6200	.1641	48	57	.1584	48	56	.1528	47	52	.1476	46	53	.1423	45	49
6300	.1689	49	57	.1632	48	57	.1575	48	53	.1522	46	54	.1468	46	50
6400	.1738	50	58	.1680	49	57	.1623	48	55	.1568	47	54	.1514	48	51
6500	.1788	50	59	.1729	50	58	.1671	49	56	.1615	48	53	.1562	47	53
6600	.1838	51	59	.1779	50	59	.1720	50	57	.1663	49	54	.1609	47	54
6700	.1889	53	60	.1829	51	59	.1770	50	58	.1712	49	56	.1656	47	54
6800	.1942	54	62	.1880	52	60	.1820	51	59	.1761	50	58	.1703	48	54

TABLE 1.—Continued.—*Auxiliary m.*

z	1200	Δ_s	Δ_v	1250	Δ_s	Δ_v	1300	Δ_s	Δ_v	1350	Δ_s	Δ_v	1400	Δ_s	Δ_v
400	.0188	50	14	.0174	46	13	.0161	43	12	.0149	41	10	.0139	38	10
500	.0238	51	18	.0220	47	16	.0204	44	14	.0190	41	13	.0177	38	13
600	.0289	52	22	.0267	49	19	.0248	45	17	.0231	42	16	.0215	39	15
700	.0341	54	25	.0316	50	23	.0293	47	20	.0273	43	19	.0254	41	17
800	.0395	55	29	.0366	51	26	.0340	48	24	.0316	45	21	.0295	42	21
900	.0450	57	33	.0417	52	29	.0388	49	27	.0361	46	24	.0337	43	24
1000	.0507	57	38	.0469	55	32	.0437	51	30	.0407	48	27	.0380	44	26
1100	.0564	58	40	.0524	55	36	.0488	52	33	.0455	48	31	.0424	45	29
1200	.0622	59	43	.0579	56	39	.0540	52	37	.0503	49	34	.0469	46	32
1300	.0681	61	46	.0635	57	43	.0592	53	40	.0552	50	37	.0515	47	34
1400	.0742	62	50	.0692	58	47	.0645	55	43	.0602	52	40	.0562	49	37
1500	.0804	62	54	.0750	60	50	.0700	56	46	.0654	53	43	.0611	50	40
1600	.0866	63	56	.0810	60	54	.0756	58	49	.0707	55	46	.0661	51	45
1700	.0929	65	59	.0870	61	56	.0814	58	52	.0762	55	50	.0712	52	45
1800	.0994	66	63	.0931	63	59	.0872	59	55	.0817	57	53	.0764	54	48
1900	.1060	66	66	.0994	63	63	.0931	61	57	.0874	57	56	.0818	55	51
2000	.1126	67	69	.1057	64	65	.0992	62	61	.0931	59	58	.0873	56	55
2100	.1193	69	72	.1121	66	67	.1054	62	64	.0990	60	61	.0929	57	57
2200	.1262	70	75	.1187	66	71	.1116	63	66	.1050	61	64	.0986	58	59
2300	.1332	70	79	.1253	68	74	.1179	65	68	.1111	62	67	.1044	60	66
2400	.1402	72	81	.1321	68	77	.1244	66	71	.1173	63	69	.1104	60	66
2500	.1474	72	85	.1389	69	79	.1310	66	74	.1236	64	72	.1164	61	68
2600	.1546	73	88	.1458	71	82	.1376	67	76	.1300	64	75	.1225	62	70
2700	.1619	75	90	.1529	71	86	.1443	69	79	.1364	66	77	.1287	63	73
2800	.1694	76	94	.1600	73	88	.1512	70	82	.1430	68	80	.1350	65	76
2900	.1770	76	97	.1673	73	91	.1582	70	84	.1498	68	83	.1415	65	78
3000	.1846	77	100	.1746	75	94	.1652	71	86	.1566	69	86	.1480	67	80
3100	.1923	79	102	.1821	75	98	.1723	73	88	.1635	69	88	.1547	68	83
3200	.2002	80	106	.1896	77	100	.1796	74	92	.1704	71	89	.1615	68	86
3300	.2082	81	109	.1973	77	103	.1870	76	95	.1775	72	92	.1683	69	88
3400	.2163	82	113	.2050	80	104	.1946	76	99	.1847	73	95	.1752	70	90
3500	.2245	83	115	.2130	81	108	.2022	77	102	.1920	74	98	.1822	71	93
3600	.2328	84	117	.2211	81	112	.2099	77	105	.1994	75	101	.1893	73	95

TABLE I.—Continued.—*Auxiliary m.*

z	1450	Δ_z	Δ_v	1500	Δ_z	Δ_v	1550	Δ_z	Δ_v	1600	Δ_z	Δ_v	1650	Δ_z	Δ_v
400	.0129	35	8	.0121	33	8	.0113	30	7	.0106	29	6	.0100	27	6
500	.0164	36	10	.0154	33	11	.0143	32	8	.0135	29	8	.0127	27	8
600	.0200	37	13	.0187	34	12	.0175	32	11	.0164	31	10	.0154	29	8
700	.0237	37	16	.0221	36	14	.0207	34	12	.0195	31	12	.0183	30	10
800	.0274	39	17	.0257	36	16	.0241	34	15	.0226	32	13	.0213	30	13
900	.0313	41	20	.0293	38	18	.0275	35	17	.0258	33	15	.0243	31	14
1000	.0354	41	23	.0331	38	21	.0310	36	19	.0291	34	17	.0274	31	16
1100	.0395	42	26	.0369	40	23	.0346	37	21	.0325	35	20	.0305	33	18
1200	.0437	44	28	.0409	40	26	.0383	39	23	.0360	36	22	.0338	33	20
1300	.0481	44	32	.0449	42	27	.0422	39	26	.0396	36	25	.0371	35	21
1400	.0525	46	34	.0491	43	30	.0461	40	29	.0432	38	26	.0406	36	23
1500	.0571	48	37	.0534	44	33	.0501	41	31	.0470	39	28	.0442	37	25
1600	.0619	48	41	.0578	46	36	.0542	43	33	.0509	40	30	.0479	38	28
1700	.0667	49	43	.0624	46	39	.0585	43	36	.0549	41	32	.0517	38	30
1800	.0716	51	46	.0670	48	42	.0628	46	38	.0590	43	35	.0555	40	32
1900	.0767	51	49	.0718	49	44	.0674	46	41	.0633	43	38	.0595	41	34
2000	.0818	54	51	.0767	50	47	.0720	47	44	.0676	45	40	.0636	43	37
2100	.0872	55	55	.0817	52	50	.0767	48	46	.0721	46	42	.0679	43	40
2200	.0927	55	58	.0869	53	54	.0815	50	48	.0767	48	45	.0722	45	42
2300	.0982	56	60	.0922	53	57	.0865	51	50	.0815	48	48	.0767	46	45
2400	.1038	58	63	.0975	55	59	.0916	53	53	.0863	49	50	.0813	46	47
2500	.1096	59	66	.1030	56	61	.0969	53	57	.0912	50	53	.0859	47	49
2600	.1155	59	69	.1086	57	64	.1022	55	60	.0962	52	56	.0906	50	51
2700	.1214	60	71	.1143	58	66	.1077	55	63	.1014	53	58	.0956	50	54
2800	.1274	63	73	.1201	59	69	.1132	57	65	.1067	55	61	.1006	52	56
2900	.1337	63	77	.1260	61	71	.1189	58	67	.1122	55	64	.1058	53	58
3000	.1400	64	79	.1321	62	74	.1247	59	70	.1177	56	66	.1111	54	60
3100	.1464	65	81	.1383	62	77	.1306	60	73	.1233	58	68	.1165	56	63
3200	.1529	66	84	.1445	64	79	.1366	61	75	.1291	59	70	.1221	57	67
3300	.1595	67	86	.1509	64	82	.1427	62	77	.1350	59	72	.1278	57	70
3400	.1662	67	89	.1573	65	84	.1489	63	80	.1409	61	74	.1335	58	71
3500	.1729	69	91	.1638	67	86	.1552	64	82	.1470	62	77	.1393	60	73
3600	.1798	69	93	.1705	67	89	.1616	65	84	.1532	63	79	.1453	60	76

TABLE I.—Continued.—*Auxiliary m.*

z	1700	Δ_z	Δ_v	1750	Δ_z	Δ_v	1800	Δ_z	Δ_v	1850	Δ_z	Δ_v	1900	Δ_z	Δ_v
400	.0094	25	6	.0088	25	4	.0084	23	5	.0079	22	3	.0076	20	4
500	.0119	27	6	.0113	25	6	.0107	23	6	.0101	22	5	.0096	21	5
600	.0146	27	8	.0138	25	8	.0130	24	7	.0123	22	6	.0117	21	6
700	.0173	27	10	.0163	25	9	.0154	24	9	.0145	23	7	.0138	22	7
800	.0200	29	12	.0188	27	10	.0178	25	10	.0168	24	8	.0160	22	8
900	.0229	29	14	.0215	28	12	.0203	26	11	.0192	25	10	.0182	24	10
1000	.0258	29	15	.0243	28	14	.0229	27	12	.0217	26	11	.0206	24	10
1100	.0287	31	16	.0271	30	15	.0256	28	13	.0243	26	13	.0230	25	11
1200	.0318	32	17	.0301	30	17	.0284	28	15	.0269	26	14	.0255	25	13
1300	.0350	33	19	.0331	31	19	.0312	29	17	.0295	28	15	.0280	26	14
1400	.0383	34	21	.0362	31	21	.0341	30	18	.0323	28	17	.0306	27	15
1500	.0417	34	24	.0393	32	22	.0371	31	20	.0351	30	18	.0333	28	17
1600	.0451	36	26	.0425	34	23	.0402	31	21	.0381	30	20	.0361	28	18
1700	.0487	36	28	.0459	34	26	.0433	33	22	.0411	30	22	.0389	30	19
1800	.0523	38	30	.0493	36	27	.0466	34	25	.0441	32	22	.0419	30	22
1900	.0561	38	32	.0529	37	29	.0500	35	27	.0473	33	24	.0449	30	23
2000	.0599	40	33	.0566	37	31	.0535	35	29	.0506	34	27	.0479	32	24
2100	.0639	41	36	.0603	39	33	.0570	36	30	.0540	35	29	.0511	33	25
2200	.0680	42	38	.0642	40	36	.0606	38	31	.0575	35	31	.0544	34	27
2300	.0722	44	40	.0682	41	38	.0644	39	34	.0610	37	32	.0578	35	29
2400	.0766	44	43	.0723	42	40	.0683	40	36	.0647	38	34	.0613	36	31
2500	.0810	45	45	.0765	43	42	.0723	41	38	.0685	38	36	.0649	37	33
2600	.0855	47	47	.0808	44	44	.0764	42	41	.0723	40	37	.0686	38	35
2700	.0902	48	50	.0852	46	46	.0806	43	43	.0763	41	39	.0724	38	37
2800	.0950	50	52	.0898	47	49	.0849	44	45	.0804	42	42	.0762	40	38
2900	.1000	51	55	.0945	48	52	.0893	46	47	.0846	44	44	.0802	41	40
3000	.1051	51	58	.0993	49	54	.0939	47	49	.0890	44	47	.0843	42	42
3100	.1102	52	60	.1042	50	56	.0986	48	52	.0934	46	49	.0885	44	44
3200	.1154	54	62	.1092	52	58	.1034	49	54	.0980	47	51	.0929	44	47
3300	.1208	56	64	.1144	52	61	.1083	50	56	.1027	47	53	.0973	46	49
3400	.1264	56	68	.1196	54	63	.1133	51	59	.1074	49	55	.1019	47	51
3500	.1320	57	70	.1250	55	66	.1184	53	61	.1123	50	57	.1066	48	53
3600	.1377	58	72	.1305	55	68	.1237	54	64	.1173	51	59	.1114	50	55

TABLE I.—Continued.—*Auxiliary m.*

z	1950	Δ_z	Δ_v	2000	Δ_z	Δ_v	2050	Δ_z	Δ_v	2100	Δ_z	Δ_v	2150	Δ_z	Δ_v
400	.0072	19	2	.0070	19	5	.0065	17	3	0062	16	3	.0059	16	3
500	.0091	20	2	.0089	18	7	.0082	18	4	.0078	17	3	.0075	16	4
600	.0111	20	4	.0107	18	7	.0100	18	5	.0095	18	4	.0091	16	4
700	.0131	21	6	.0125	20	7	.0118	19	5	.0113	18	6	.0107	18	4
800	.0152	22	7	.0145	20	8	.0137	20	6	.0131	19	6	.0125	18	6
900	.0174	22	9	.0165	21	8	.0157	20	7	.0150	19	7	.0143	18	7
1000	.0196	23	10	.0186	22	9	.0177	21	8	.0169	20	8	.0161	19	8
1100	.0219	23	11	.0208	22	10	.0198	21	9	.0189	20	9	.0180	19	9
1200	.0242	24	12	.0230	23	11	.0219	22	10	.0209	21	10	.0199	20	9
1300	.0266	25	13	.0253	23	12	.0241	22	11	.0230	21	11	.0219	20	10
1400	.0291	25	15	.0276	25	13	.0263	23	12	.0251	22	12	.0239	22	11
1500	.0316	27	15	.0301	25	15	.0286	24	13	.0273	22	12	.0261	21	13
1600	.0343	27	17	.0326	26	16	.0310	24	15	.0295	24	13	.0282	22	13
1700	.0370	27	18	.0352	26	18	.0334	25	15	.0319	24	15	.0304	22	14
1800	.0397	29	19	.0378	27	19	.0359	26	16	.0343	25	17	.0326	24	14
1900	.0426	29	21	.0405	28	20	.0385	27	17	.0368	25	18	.0350	24	15
2000	.0455	31	22	.0433	29	21	.0412	28	19	.0393	26	19	.0374	25	16
2100	.0486	31	24	.c462	29	22	.0440	28	21	.0419	27	20	.0399	26	17
2200	.0517	32	26	.0491	31	23	.0468	29	22	.0446	28	21	.0425	26	19
2300	.0549	33	27	.0522	32	25	.0497	29	23	.0474	28	23	.0451	27	20
2400	.0582	34	28	.0554	32	28	.0526	31	24	.0502	29	24	.0478	28	21
2500	.0616	35	30	.0586	33	29	.0557	32	26	.0531	30	25	.0506	29	22
2600	.0651	36	32	.0619	34	30	.0589	33	28	.0561	31	26	.0535	30	24
2700	.0687	37	34	.0653	35	31	.0622	33	30	.0592	32	27	.0565	30	25
2800	.0724	38	36	.0688	36	33	.0655	35	31	.0624	32	29	.0595	31	26
2900	.0762	39	38	.0724	37	34	.0690	35	34	.0656	34	30	.0626	32	27
3000	.0801	40	40	.0761	39	36	.0725	37	35	.0690	35	32	.0658	34	29
3100	.0841	41	41	.0800	39	38	.0762	37	37	.0725	36	33	.0692	34	31
3200	.0882	42	43	.0839	41	40	.0799	38	38	.0761	37	35	.0726	35	33
3300	.0924	44	44	.0880	41	42	.0838	39	40	.0798	38	37	.0761	36	34
3400	.0968	45	47	.0921	43	44	.0877	41	41	.0836	39	39	.0797	37	36
3500	.1013	46	50	.0963	44	45	.0918	41	43	.0875	39	41	.0834	38	37
3600	.1059	47	53	.1007	46	48	.0959	43	45	.0914	41	42	.0872	39	39

TABLE I.—CONTINUED.—*Auxiliary m.*

z	1200	Δ_z	Δ_v	1250	Δ_z	Δ_v	1300	Δ_z	Δ_v	1350	Δ_z	Δ_v	1400	Δ_z	Δ_v
3600	.2328	84	117	.2211	81	112	.2099	77	105	.1994	75	101	.1893	73	95
3700	.2412	85	120	.2292	81	116	.2176	79	107	.2069	76	103	.1966	73	99
3800	.2497	87	124	.2373	83	118	.2255	80	110	.2145	77	106	.2039	74	101
3900	.2584	87	128	.2456	83	121	.2335	81	113	.2222	79	109	.2113	76	103
4000	.2671	90	132	.2539	85	123	.2416	82	115	.2301	79	112	.2189	77	106
4100	.2761	90	137	.2624	86	126	.2498	83	118	.2380	80	114	.2266	78	109
4200	.2851	90	141	.2710	88	129	.2581	84	121	.2460	81	116	.2344	79	112
4300	.2941	92	143	.2798	88	133	.2665	85	124	.2541	83	118	.2423	79	114
4400	.3033	94	147	.2886	89	136	.2750	87	126	.2624	84	122	.2502	81	116
4500	.3127	94	152	.2975	91	138	.2837	87	129	.2708	85	125	.2583	82	120
4600	.3221	96	155	.3066	93	142	.2924	89	131	.2793	86	128	.2665	83	123
4700	.3317	96	158	.3159	94	146	.3013	90	134	.2879	86	131	.2748	85	125
4800	.3413	98	160	.3253	94	150	.3103	91	138	.2965	88	132	.2833	85	129
4900	.3511	99	164	.3347	95	153	.3194	92	141	.3053	89	135	.2918	86	131
5000	.3610	101	168	.3442	97	156	.3286	93	144	.3142	91	138	.3004	87	134
5100	.3711	102	172	.3539	98	160	.3379	95	146	.3233	91	142	.3091	88	136
5200	.3813	102	176	.3637	99	163	.3474	96	150	.3324	93	145	.3179	90	138
5300	.3915	104	179	.3736	100	166	.3570	97	153	.3417	94	148	.3269	91	141
5400	.4019	105	183	.3836	101	169	.3667	98	156	.3511	95	151	.3360	92	144
5500	.4124	107	187	.3937	103	172	.3765	99	159	.3606	96	154	.3452	94	146
5600	.4231	107	191	.4040	104	176	.3864	101	162	.3702	97	156	.3546	94	150
5700	.4338	110	194	.4144	105	179	.3965	102	166	.3799	98	159	.3640	95	153
5800	.4448	111	199	.4249	107	183	.4067	103	170	.3897	100	162	.3735	97	156
5900	.4559	112	203	.4356	108	186	.4170	104	173	.3997	102	165	.3832	98	159
6000	.4671	113	207	.4464	109	190	.4274	105	175	.4099	102	169	.3930	99	162
6100	.4784	113	211	.4573	110	194	.4379	107	178	.4201	103	172	.4029	100	165
6200	.4897	116	214	.4683	111	197	.4486	108	182	.4304	105	175	.4129	102	168
6300	.5013	118	219	.4794	113	200	.4594	109	185	.4409	106	178	.4231	102	171
6400	.5131	119	224	.4907	114	203	.4703	110	188	.4515	107	182	.4333	104	173
6500	.5250	120	229	.5021	115	208	.4813	112	191	.4622	108	185	.4437	105	177
6600	.5370	120	234	.5136	117	211	.4925	113	195	.4730	109	188	.4542	106	180
6700	.5490	122	237	.5253	120	215	.5038	115	199	.4839	111	191	.4648	108	183
6800	.5612	125	239	.5373	120	220	.5153	117	203	.4950	112	194	.4756	109	186

TABLE I.—Continued.—*Auxiliary m.*

z	1450	Δ_1	Δ_2	1500	Δ_1	Δ_2	1550	Δ_1	Δ_2	1600	Δ_1	Δ_2	1650	Δ_1	Δ_2
3600	.1798	69	93	.1705	67	89	.1616	65	84	.1532	63	79	.1453	60	76
3700	.1867	71	95	.1772	69	91	.1681	67	86	.1595	64	82	.1513	62	78
3800	.1938	72	97	.1841	70	93	.1748	68	89	.1659	66	84	.1575	63	81
3900	.2010	73	99	.1911	70	95	.1816	68	91	.1725	66	87	.1638	64	83
4000	.2083	74	102	.1981	72	97	.1884	69	93	.1791	67	89	.1702	65	85
4100	.2157	75	104	.2053	73	100	.1953	70	95	.1858	68	91	.1767	66	87
4200	.2232	77	106	.2126	73	103	.2023	71	97	.1926	69	93	.1833	67	89
4300	.2309	77	110	.2199	75	105	.2094	73	99	.1995	70	95	.1900	67	91
4400	.2386	77	112	.2274	75	107	.2167	74	102	.2065	72	98	.1967	69	92
4500	.2463	79	114	.2349	76	108	.2241	74	104	.2137	72	101	.2036	70	95
4600	.2542	81	117	.2425	78	110	.2315	75	106	.2209	73	103	.2106	71	97
4700	.2623	81	120	.2503	79	113	.2390	76	108	.2282	74	105	.2177	72	100
4800	.2704	83	122	.2582	79	116	.2466	77	110	.2356	75	107	.2249	73	102
4900	.2787	83	126	.2661	81	118	.2543	79	112	.2431	76	109	.2322	73	104
5000	.2870	85	128	.2742	82	120	.2622	79	115	.2507	77	112	.2395	75	105
5100	.2955	86	131	.2824	84	123	.2701	81	117	.2584	78	114	.2470	76	108
5200	.3041	87	133	.2908	85	126	.2782	82	120	.2662	79	116	.2546	77	110
5300	.3128	88	135	.2993	85	129	.2864	83	123	.2741	80	118	.2623	78	112
5400	.3216	90	138	.3078	87	131	.2947	84	126	.2821	82	120	.2701	79	115
5500	.3306	90	141	.3165	88	134	.3031	85	128	.2903	83	123	.2780	80	117
5600	.3396	91	143	.3253	88	137	.3116	85	130	.2986	83	126	.2860	82	119
5700	.3487	92	146	.3341	89	140	.3201	87	132	.3069	85	127	.2942	82	121
5800	.3579	94	149	.3430	91	142	.3288	89	134	.3154	85	130	.3024	83	123
5900	.3673	95	152	.3521	92	144	.3377	90	138	.3239	87	132	.3107	85	125
6000	3768	96	155	.3613	94	146	.3467	91	141	.3326	88	134	.3192	86	127
6100	.3864	97	157	.3707	95	149	.3558	92	144	.3414	90	136	.3278	87	130
6200	.3961	99	159	.3802	96	152	.3650	93	146	.3504	90	139	.3365	88	132
6300	.4060	100	162	.3898	97	155	.3743	94	149	.3594	92	141	.3453	89	136
6400	.4160	100	165	.3995	98	158	.3837	95	151	.3686	93	144	.3542	90	138
6500	.4260	102	167	.4093	99	161	.3932	96	153	.3779	94	147	.3632	91	139
6600	.4362	103	170	.4192	100	164	.4028	98	155	.3873	95	150	.3723	93	141
6700	.4465	105	173	.4292	102	166	.4126	99	158	.3968	96	152	.3816	93	145
6800	.4570	106	176	.4394	102	169	.4225	100	161	.4064	97	155	.3909	95	147

TABLE I.—CONTINUED.—*Auxiliary m.*

z	1700	Δ_z	Δ_v	1750	Δ_z	Δ_v	1800	Δ_z	Δ_v	1850	Δ_z	Δ_v	1900	Δ_z	Δ_v
3600	.1377	58	72	.1305	55	68	.1237	54	64	.1173	51	59	.1114	50	55
3700	.1435	59	75	.1360	58	69	.1291	55	67	.1224	53	60	.1164	50	58
3800	.1494	61	76	.1418	59	72	.1346	56	69	.1277	54	63	.1214	52	59
3900	.1555	62	78	.1477	60	75	.1402	58	71	.1331	55	65	.1266	53	61
4000	.1617	63	80	.1537	61	77	.1460	58	74	.1386	57	67	.1319	54	63
4100	.1680	64	82	.1598	61	80	.1518	60	75	.1443	58	70	.1373	56	66
4200	.1744	65	85	.1659	62	81	.1578	61	77	.1501	59	72	.1429	56	69
4300	.1809	66	88	.1721	64	82	.1639	62	79	.1560	60	75	.1485	58	70
4400	.1875	66	90	.1785	65	84	.1701	62	81	.1620	61	77	.1543	59	73
4500	.1941	68	91	.1850	66	87	.1763	63	82	.1681	61	79	.1602	60	75
4600	.2009	68	93	.1916	67	90	.1826	65	84	.1742	63	80	.1662	60	77
4700	.2077	70	94	.1983	67	92	.1891	66	86	.1805	64	83	.1722	62	78
4800	.2147	71	97	.2050	69	93	.1957	67	88	.1869	64	85	.1784	63	81
4900	.2218	72	99	.2119	69	95	.2024	68	91	.1933	66	86	.1847	64	82
5000	.2290	72	102	.2188	71	96	.2092	68	93	.1999	67	88	.1911	65	84
5100	.2362	74	103	.2259	72	99	.2160	70	94	.2066	68	90	.1976	66	86
5200	.2436	75	105	.2331	72	101	.2230	70	96	.2134	69	92	.2042	67	88
5300	.2511	75	108	.2403	74	103	.2300	72	97	.2203	69	94	.2109	68	90
5400	.2586	77	109	.2477	75	105	.2372	73	100	.2272	71	95	.2177	69	92
5500	.2663	78	111	.2552	75	107	.2445	74	102	.2343	72	97	.2246	70	94
5600	.2741	80	114	.2627	77	108	.2519	74	104	.2415	73	99	.2316	70	96
5700	.2821	80	117	.2704	78	111	.2593	76	105	.2488	73	102	.2386	72	97
5800	.2901	81	119	.2782	79	113	.2669	77	108	.2561	75	103	.2458	73	99
5900	.2982	82	121	.2861	80	115	.2746	78	110	.2636	76	105	.2531	74	101
6000	.3064	83	123	.2941	82	117	.2824	79	112	.2712	77	107	.2605	74	103
6100	.3147	85	124	.3023	82	120	.2903	80	114	.2789	78	110	.2679	76	104
6200	.3232	85	127	.3105	84	122	.2983	81	116	.2867	79	112	.2755	78	106
6300	.3317	87	128	.3189	84	125	.3064	83	118	.2946	80	113	.2833	78	109
6400	.3404	89	131	.3273	85	126	.3147	83	121	.3026	81	115	.2911	79	111
6500	.3493	89	135	.3358	87	128	.3230	85	123	.3107	83	117	.2990	80	112
6600	.3582	89	137	.3445	87	130	.3315	85	125	.3190	84	120	.3070	81	114
6700	.3671	91	139	.3532	89	132	.3400	87	126	.3274	84	123	.3151	83	116
6800	.3762	92	141	.3621	89	134	.3487	87	129	.3358	86	124	.3234	84	118

TABLE 1.—Continued.—*Auxiliary m.*

z	1950	Δ_s	Δ_v	2000	Δ_s	Δ_v	2050	Δ_s	Δ_v	2100	Δ_s	Δ_v	2150	Δ_s	Δ_v
3600	.1059	47	52	.1007	46	48	.0959	43	45	.0914	41	42	.0872	39	39
3700	.1106	49	53	.1053	46	51	.1002	44	47	.0955	42	44	.0911	40	41
3800	.1155	50	56	.1099	48	53	.1046	46	49	.0997	43	46	.0951	42	42
3900	.1205	51	58	.1147	48	55	.1092	46	52	.1040	45	47	.0993	43	44
4000	.1256	51	61	.1195	49	57	.1138	48	53	.1085	46	49	.1036	44	46
4100	.1307	53	63	.1244	51	58	.1186	49	55	.1131	47	51	.1080	45	48
4200	.1360	55	65	.1295	52	60	.1235	50	57	.1178	48	53	.1125	46	50
4300	.1415	55	68	.1347	54	62	.1285	51	59	.1226	49	55	.1171	47	52
4400	.1470	57	69	.1401	55	65	.1336	53	61	.1275	50	57	.1218	48	54
4500	.1527	58	71	.1456	56	67	.1389	54	64	.1325	52	59	.1266	50	56
4600	.1585	59	73	.1512	57	69	.1443	55	66	.1377	54	61	.1316	51	58
4700	.1644	59	75	.1569	58	71	.1498	56	67	.1431	54	64	.1367	52	60
4800	.1703	62	76	.1627	59	73	.1554	57	69	.1485	56	66	.1419	54	62
4900	.1765	62	79	.1686	60	75	.1611	58	70	.1541	56	68	.1473	54	65
5000	.1827	63	81	.1746	62	77	.1669	60	72	.1597	58	70	.1527	56	66
5100	.1890	64	82	.1808	62	79	.1729	61	74	.1655	58	72	.1583	57	68
5200	.1954	65	84	.1870	63	80	.1790	61	77	.1713	59	73	.1640	58	70
5300	.2019	66	86	.1933	64	82	.1851	63	79	.1772	61	74	.1698	59	71
5400	.2085	67	88	.1997	66	83	.1914	63	81	.1833	62	76	.1757	60	73
5500	.2152	68	89	.2063	66	86	.1977	64	82	.1895	63	78	.1817	61	74
5600	.2220	69	91	.2129	67	88	.2041	66	83	.1958	63	80	.1878	62	76
5700	.2289	70	93	.2196	68	89	.2107	67	86	.2021	65	81	.1940	63	77
5800	.2359	71	95	.2264	69	90	.2174	67	88	.2086	66	83	.2003	65	79
5900	.2430	72	97	.2333	70	92	.2241	69	89	.2152	67	84	.2068	65	81
6000	.2502	73	99	.2403	72	93	.2310	69	91	.2219	68	86	.2133	66	83
6100	.2575	74	100	.2475	72	96	.2379	70	92	.2287	70	88	.2199	67	84
6200	.2649	75	102	.2547	74	98	.2449	72	92	.2357	70	91	.2266	68	86
6300	.2724	76	103	.2621	74	100	.2521	73	94	.2427	70	93	.2334	69	88
6400	.2800	78	105	.2695	76	101	.2594	73	97	.2497	71	94	.2403	71	89
6500	.2878	78	107	.2771	76	104	.2667	75	99	.2568	73	94	.2474	71	91
6600	.2956	79	109	.2847	78	105	.2742	76	101	.2641	74	96	.2545	72	92
6700	.3035	81	110	.2925	78	107	.2818	77	103	.2715	75	98	.2617	73	93
6800	.3116	82	113	.3003	80	108	.2895	78	105	.2790	76	100	.2690	74	95

TABLE II.

For Spherical Projectiles.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
2000	0	25	0.00	1	.00000	40	.000	12
1990	25	24	0.01	1	.00040	40	.012	13
1980	49	25	0.02	2	.00080	41	.025	12
1970	74	25	0.04	4	.00121	42	.037	13
1960	99	25	0.08	5	.00163	42	.050	13
1950	124	26	0.13	5	.00205	43	.063	13
1940	150	25	0.18	7	.00248	44	.076	13
1930	175	26	0.25	8	.00292	44	.089	13
1920	201	25	0.33	9	.00336	45	.102	14
1910	226	26	0.42	11	.00381	46	.116	13
1900	252	26	0.53	12	.00427	46	.129	14
1890	278	26	0.65	13	.00473	47	.143	14
1880	304	26	0.78	14	.00520	48	.157	14
1870	330	27	0.92	15	.00568	49	.171	14
1860	357	26	1.07	17	.00617	49	.185	14
1850	383	26	1.24	19	.00666	50	.199	15
1840	409	27	1.43	20	.00716	51	.214	14
1830	436	27	1.63	21	.00767	52	.228	15
1820	463	27	1.84	23	.00819	53	.243	15
1810	490	27	2.07	24	.00872	54	.258	15
1800	517	28	2.31	26	.00926	55	.273	15
1790	545	27	2.57	27	.00981	55	.288	16
1780	572	28	2.84	30	.01036	57	.304	15
1770	600	28	3.14	31	.01093	57	.319	16
1760	628	28	3.45	33	.01150	59	.335	16
1750	656	28	3.78	35	.01209	59	.351	16
1740	684	28	4.13	37	.01268	61	.367	16
1730	712	29	4.50	39	.01329	61	.383	17
1720	741	28	4.89	41	.01390	63	.400	16
1710	769	29	5.30	43	.01453	64	.416	17

TABLE II.—Continued

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
1700	798	29	5.73	45	.01517	65	.433	17
1690	827	29	6.18	47	.01582	66	.450	18
1680	856	30	6.65	50	.01648	67	.468	17
1670	886	29	7.15	52	.01715	68	.485	18
1660	915	30	7.67	54	.01783	70	.503	18
1650	945	30	8.21	56	.01853	71	.521	18
1640	975	30	8.77	58	.01924	72	.539	19
1630	1005	31	9.35	62	.01996	74	.558	18
1620	1036	30	9.97	64	.02070	75	.576	19
1610	1066	30	10.61	66	.02145	77	.595	19
1600	1096	31	11.27	69	.02222	78	.614	19
1590	1127	31	11.96	72	.02300	79	.633	20
1580	1158	31	12.68	76	.02379	81	.653	20
1570	1189	31	13.44	78	.02460	82	.673	20
1560	1220	32	14.22	82	.02542	84	.693	20
1550	1252	32	15.04	86	.02626	86	.713	21
1540	1284	32	15.90	88	.02712	87	.734	21
1530	1316	32	16.78	92	.02799	89	.755	21
1520	1348	32	17.70	95	.02888	91	.776	21
1510	1380	33	18.65	98	.02979	93	.797	22
1500	1413	33	19.63	100	.03072	94	.819	22
1490	1446	33	20.63	105	.03166	96	.841	22
1480	1479	33	21.68	109	.03262	98	.863	22
1470	1512	34	22.77	114	.03360	101	.885	23
1460	1546	34	23.91	119	.03461	103	.908	23
1450	1580	34	25.10	124	.03564	105	.931	24
1440	1614	34	26.34	128	.03669	107	.955	24
1430	1648	34	27.62	133	.03776	109	.979	24
1420	1682	35	28.95	138	.03885	112	1.003	25
1410	1717	35	30.33	143	.03997	114	1.028	25
1400	1752	35	31.76	149	.04111	116	1.053	26
1390	1787	36	33.25	154	.04227	119	1.079	26
1380	1823	35	34.79	160	.04346	122	1.105	26

TABLE II.—Continued.

u	$S(v)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
1370	1858	36	36.39	164	.04468	124	1.131	27
1360	1894	37	38.03	170	.04592	127	1.158	27
1350	1931	36	39.73	175	.04719	129	1.185	27
1340	1967	37	41.48	181	.04848	133	1.212	27
1330	2004	37	43.29	185	.04981	136	1.239	28
1320	2041	37	45.14	191	.05117	139	1.267	27
1310	2078	38	47.05	196	.05256	142	1.294	28
1300	2116	38	49.01	203	.05398	144	1.322	29
1290	2154	38	51.04	212	.05542	148	1.351	39
1280	2192	39	53.16	221	.05690	152	1.381	30
1270	2231	38	55.37	230	.05842	156	1.411	31
1260	2269	39	57.67	240	.05998	160	1.442	31
1250	2308	40	60.07	249	.06158	165	1.473	32
1240	2348	40	62.56	258	.06323	169	1.505	33
1230	2388	40	65.14	267	.06492	174	1.538	33
1220	2428	42	67.81	278	.06666	180	1.571	34
1210	2470	42	70.59	295	.06846	187	1.605	35
1200	2512	22	73.54	156	.07033	97	1.640	18
1195	2534	22	75.10	160	.07130	99	1.658	18
1190	2556	22	76.70	162	.07229	100	1.676	18
1185	2578	22	78.32	165	.07329	102	1.694	18
1180	2600	23	79.97	169	.07431	104	1.712	19
1175	2623	23	81.66	173	.07535	106	1.731	29
1170	2646	23	83.39	177	.07641	108	1.751	19
1165	2669	23	85.16	182	.07749	110	1.770	20
1160	2692	23	86.98	186	.07859	113	1.790	20
1155	2715	24	88.84	190	.07972	115	1.810	21
1150	2739	24	90.74	195	.08087	117	1.831	21
1145	2763	24	92.69	199	.08204	120	1.852	21
1140	2787	25	94.68	205	.08324	122	1.873	22
1135	2812	25	96.73	209	.08446	124	1.895	22
1130	2837	24	98.82	215	.08570	127	1.917	23
1125	2861	25	00.97	221	.08697	130	1.940	23

TABLE II.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
1120	2886	26	103.18	226	.08827	132	1.963	23
1115	2912	26	105.44	233	.08959	135	1.986	23
1110	2938	26	107.77	239	.09094	138	2.009	24
1105	2964	27	110.16	246	.09232	141	2.033	24
1100	2991	26	112.62	251	.09373	143	2.057	24
1095	3017	27	115.13	259	.09516	147	2.081	25
1090	3044	27	117.72	266	.09663	149	2.106	26
1085	3071	28	120.38	275	.09812	153	2.132	26
1080	3099	28	123.13	283	.09965	156	2.158	26
1075	3127	28	125.96	291	.10121	159	2.184	26
1070	3155	29	128.87	300	.10280	163	2.210	27
1065	3184	29	131.87	308	.10443	166	2.237	28
1060	3213	30	134.95	317	.10609	170	2.265	28
1055	3243	30	138.12	326	.10779	173	2.293	28
1050	3273	30	141.38	338	.10952	177	2.321	29
1045	3303	30	144.76	346	.11129	181	2.350	29
1040	3333	31	148.22	355	.11310	185	2.379	30
1035	3364	31	151.77	364	.11495	189	2.409	31
1030	3395	32	155.41	374	.11684	193	2.440	31
1025	3427	32	159.15	384	.11877	197	2.471	31
1020	3459	32	162.99	394	.12074	202	2.502	32
1015	3491	33	166.93	406	.12276	206	2.534	32
1010	3524	33	170.99	418	.12482	211	2.566	33
1005	3557	34	175.17	430	.12693	215	2.599	33
1000	3591	34	179.47	443	.12908	220	2.632	33
995	3625	35	183.90	456	.13128	226	2.665	34
990	3660	35	188.46	470	.13354	231	2.699	35
985	3695	36	193.16	484	.13585	236	2.734	36
980	3731	36	198.00	498	.13821	241	2.770	36
975	3767	36	202.98	513	.14062	246	2.806	37
970	3803	37	208.11	529	.14308	252	2.843	38
965	3840	37	213.40	546	.14560	258	2.881	39
960	3877	38	218.86	563	.14818	264	2.920	39

TABLE II.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
955	3915	38	224.49	580	.15082	270	2.959	40
950	3953	39	230.29	600	.15352	276	2.999	41
945	3992	39	236.29	620	.15628	283	3.040	42
940	4031	39	242.49	637	.15911	290	3.082	43
935	4070	40	248.86	657	.16201	297	3.125	43
930	4110	40	255.43	676	.16498	304	3.168	44
925	4151	41	262.19	698	.16802	311	3.212	45
920	4192	42	269.17	720	.17113	319	3.257	46
915	4234	43	276.37	743	.17432	327	3.303	47
910	4277	43	283.80	767	.17759	335	3.350	47
905	4320	43	291.47	793	.18094	343	3.397	48
900	4363	44	299.40	819	.18437	352	3.445	49
895	4407	44	307.59	845	.18789	360	3.494	50
890	4451	45	316.04	873	.19149	369	3.544	51
885	4496	46	324.77	901	.19518	378	3.595	52
880	4542	47	333.78	928	.19896	387	3.647	53
875	4589	47	343.06	961	.20283	397	3.700	54
870	4636	48	352.67	997	.20680	407	3.754	55
865	4684	48	362.64	1032	.21087	418	3.809	56
860	4732	49	372.96	1064	.21505	428	3.865	57
855	4781	49	383.60	1099	.21933	439	3.922	58
850	4830	50	394.59	1137	.22372	451	3.980	59
845	4880	51	405.96	1175	.22823	462	4.039	61
840	4931	52	417.71	1216	.23285	476	4.100	61
835	4983	53	429.87	1258	.23761	487	4.161	63
830	5036	53	442.45	1302	.24248	498	4.224	64
825	5089	54	455.47	1347	.24746	511	4.288	66
820	5143	55	468.94	1395	.25257	526	4.354	67
815	5198	55	482.89	1444	.25783	540	4.421	68
810	5253	56	497.33	1495	.26323	553	4.489	70
805	5309	57	512.28	1549	.26876	568	4.559	71
800	5366	58	527.77	1604	.27444	587	4.630	72
795	5424	59	543.81	1661	.28031	601	4.702	74

TABLE II.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
790	5483	59	560.42	1722	.28632	617	4.776	76
785	5542	60	577.64	1784	.29249	634	4.852	77
780	5602	61	595.48	1849	.29883	650	4.929	79
775	5663	62	613.97	1916	.30533	670	5.008	80
770	5725	63	633.13	1988	.31203	688	5.088	82
765	5788	64	653.01	2062	.31891	707	5.170	84
760	5852	65	673.63	2138	.32598	727	5.254	86
755	5917	66	695.01	2218	.33325	748	5.340	87
750	5983	67	717.19	2303	.34073	770	5.427	90
745	6050	68	740.22	2389	.34843	791	5.517	91
740	6118	69	764.11	2480	.35634	814	5.608	93
735	6187	69	788.91	2574	.36448	837	5.701	96
730	6256	71	814.65	2673	.37285	861	5.797	97
725	6327	72	841.38	2776	.38146	887	5.894	100
720	6399	73	869.14	2882	.39033	912	5.994	102
715	6472	74	897.96	2996	.39945	940	6.096	104
710	6546	75	927.92	3115	.40885	968	6.200	106
705	6621	77	959.07	3238	.41853	995	6.306	109
700	6698	78	991.45	3366	.42848	1024	6.415	111
695	6776	79	1025.2	350	.43872	1054	6.526	114
690	6855	80	1060.2	364	.44926	1089	6.640	116
685	6935	81	1096.6	378	.46015	1128	6.756	119
680	7016	82	1134.4	394	.47143	1159	6.875	122
675	7098	84	1173.8	409	.48302	1192	6.997	125
670	7182	85	1214.7	427	.49494	1228	7.122	127
665	7267	87	1257.4	444	.50722	1267	7.249	131
660	7354	88	1301.8	463	.51989	1307	7.380	134
655	7442	89	1348.1	482	.53296	1349	7.514	137
650	7531	91	1396.3	502	.54645	1392	7.651	140
645	7622	92	1446.5	523	.56037	1436	7.791	143
640	7714	94	1498.8	546	.57473	1482	7.934	147
635	7808	95	1553.4	568	.58955	1529	8.081	150
630	7903	97	1610.2	592	.60484	1579	8.231	154

TABLE II.—Continued.

u	S(u)	Diff	A(u)	Diff	I(u)	Diff	T(u)	Diff
625	8000	98	1669.4	618	.62063	1633	8.385	158
620	8098	100	1731.2	644	.63696	1690	8.543	162
615	8198	101	1795.6	673	.65386	1737	8.705	166
610	8299	103	1862.9	702	.67123	1799	8.871	170
605	8402	105	1933.1	733	.68922	1859	9.041	174
600	8507	107	2006.4	765	.70781	1923	9.215	179
595	8614	108	2082.9	800	.72704	1988	9.394	183
590	8722	111	2162.9	836	.74692	2055	9.577	188
585	8833	112	2246.5	872	.76747	2126	9.765	192
580	8945	114	2333.7	911	.78873	2199	9.957	197
575	9059	116	2424.8	954	.81072	2276	10.154	203
570	9175	118	2520.2	998	.83348	2356	10.357	208
565	9293	120	2620.0	1043	.85704	2440	10.565	213
560	9413	122	2724.3	1091	.88144	2526	10.778	219
555	9535	124	2833.4	1142	.90670	2617	10.997	225
550	9659	126	2947.6	1196	.93287	2711	11.222	231
545	9785	129	3067.2	1252	.95998	2810	11.453	237
540	9914	131	3192.4	1312	.98808	2913	11.690	243
535	10045	133	3323.6	1374	1.01721	3019	11.933	250
530	10178	135	3461.0	1440	1.04740	3133	12.183	257
525	10313	138	3605.0	1509	1.07873	3247	12.440	264
520	10451	140	3755.9	1582	1.11120	3366	12.704	271
515	10591	143	3914.1	1660	1.14486	3495	12.975	279
510	10734	146	4080.1	1743	1.17981	3633	13.254	287
505	10880	148	4254.4	1829	1.21614	3779	13.541	295
500	11028	151	4437.3	1920	1.25393	3919	13.836	302
495	11179	153	4629.3	2017	1.29312	4070	14.138	312
490	11332	156	4831.0	2118	1.33382	4232	14.450	320
485	11488	160	5042.8	2226	1.37614	4399	14.770	330
480	11648	162	5265.4	2340	1.42013	4575	15.100	340
475	11810	165	5499.4	2461	1.46588	4760	15.440	350
470	11975	168	5745.5	2588	1.51348	4953	15.790	360
465	12143	172	6004.3	2724	1.56301	5157	16.150	370

TABLE II.—CONTINUED.

u	$S(u)$	Diff	$A(u)$	Diff	$I(u)$	Diff	$T(u)$	Diff
460	12315	175	6276.7	2868	1.61458	5368	16.520	382
455	12490	178	6563.5	3020	1.66826	5593	16.902	394
450	12668		6865.5		1.72419		17.296	

TABLE III.

Values of $\frac{\delta_t}{\delta}$ for temperature and pressure of atmosphere two-thirds saturated with moisture.

F	28 in.	29 in.	30 in.	31 in.	F	28 in.	29 in.	30 in.	31 in.
0°	0.945	0.912	0.882	0.853	28°	1.004	0.969	0.937	0.907
1	0.947	0.914	0.884	0.855	29	1.006	0.971	0.939	0.909
2	0.949	0.916	0.886	0.857	30	1.008	0.973	0.941	0.911
3	0.951	0.918	0.888	0.859					
4	0.953	0.920	0.890	0.861	31	1.010	0.975	0.943	0.912
5	0.955	0.922	0.892	0.863	32	1.012	0.977	0.945	0.914
6	0.957	0.924	0.893	0.865	33	1.014	0.979	0.947	0.916
7	0.959	0.926	0.895	0.867	34	1.016	0.981	0.949	0.918
8	0.962	0.928	0.897	0.869	35	1.018	0.983	0.951	0.920
9	0.964	0.930	0.899	0.870	36	1.021	0.986	0.953	0.922
10	0.966	0.932	0.901	0.872	37	1.023	0.988	0.955	0.924
11	0.968	0.935	0.903	0.874	38	1.025	0.990	0.957	0.926
12	0.970	0.937	0.905	0.876	39	1.027	0.992	0.958	0.928
13	0.972	0.939	0.907	0.878	40	1.029	0.994	0.960	0.930
14	0.974	0.941	0.909	0.880	41	1.031	0.996	0.962	0.932
15	0.976	0.943	0.911	0.882	42	1.033	0.998	0.964	0.933
16	0.978	0.945	0.913	0.884	43	1.035	1.000	0.966	0.935
17	0.981	0.947	0.915	0.886	44	1.037	1.002	0.968	0.937
18	0.983	0.949	0.917	0.888	45	1.040	1.004	0.970	0.939
19	0.985	0.951	0.919	0.890	46	1.042	1.006	0.972	0.941
20	0.987	0.953	0.921	0.891	47	1.044	1.008	0.974	0.943
21	0.989	0.955	0.923	0.893	48	1.046	1.010	0.976	0.945
22	0.991	0.957	0.925	0.895	49	1.048	1.012	0.978	0.947
23	0.993	0.959	0.927	0.897	50	1.050	1.014	0.980	0.949
24	0.995	0.961	0.929	0.899	51	1.052	1.016	0.982	0.951
25	0.997	0.963	0.931	0.901	52	1.054	1.018	0.984	0.953
26	1.000	0.965	0.933	0.903	53	1.056	1.020	0.986	0.954
27	1.002	0.967	0.935	0.905	54	1.058	1.022	0.988	0.956

TABLE III.—Continued.

F	28 in.	29 in.	30 in.	31 in.	*F*	28 in.	29 in.	30 in.	31 in.
55°	1.061	1.024	0.990	0.958	79°	1.111	1.073	1.037	1.004
56	1.063	1.026	0.992	0.960	80	1.113	1.075	1.039	1.006
57	1.065	1.028	0.994	0.962	81	1.116	1.077	1.041	1.008
58	1.067	1.030	0.996	0.964	82	1.118	1.079	1.043	1.010
59	1.069	1.032	0.998	0.966	83	1.120	1.081	1.045	1.012
60	1.071	1.034	1.000	0.968	84	1.122	1.083	1.047	1.014
61	1.073	1.037	1.002	0.970	85	1.124	1.085	1.049	1.016
62	1.075	1.039	1.004	0.972	86	1.126	1.088	1.051	1.017
63	1.078	1.041	1.006	0.974	87	1.128	1.090	1.053	1.019
64	1.080	1.043	1.008	0.975	88	1.130	1.092	1.055	1.021
65	1.082	1.045	1.010	0.977	89	1.132	1.094	1.057	1.023
66	1.084	1.047	1.012	0.979	90	1.135	1.096	1.059	1.025
67	1.086	1.049	1.014	0.981	91	1.137	1.098	1.061	1.027
68	1.088	1.051	1.016	0.983	92	1.139	1.100	1.063	1.029
69	1.090	1.053	1.018	0.985	93	1.141	1.102	1.065	1.031
70	1.092	1.055	1.020	0.987	94	1.143	1.104	1.067	1.033
71	1.094	1.057	1.022	0.989	95	1.145	1.106	1.069	1.035
72	1.097	1.059	1.024	0.991	96	1.147	1.108	1.071	1.037
73	1.099	1.061	1.025	0.993	97	1.149	1.110	1.073	1.038
74	1.101	1.063	1.027	0.995	98	1.151	1.112	1.075	1.040
75	1.103	1.065	1.029	0.996	99	1.155	1.114	1.077	1.042
76	1.105	1.067	1.031	0.998	100	1.157	1.116	1.079	1.044
77	1.107	1.069	1.033	1.000					
78	1.109	1.071	1.035	1.002					

TABLE IV.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
0°00′	.00000	291	.00000	291	5°30′	.09644	295	.09629	294
0 10	.00291	291	.00291	291	5 40	.09939	295	.09923	293
0 20	.00582	291	.00582	291	5 50	.10234	296	.10216	294
0 30	.00873	291	.00873	291	6 00	.10530	296	.10510	295
0 40	.01164	291	.01164	291	6 10	.10826	296	.10805	294
0 50	.01455	291	.01455	291	6 20	.11122	296	.11099	295
1 00	.01746	291	.01746	291	6 30	.11418	297	.11394	294
1 10	.02037	291	.02037	291	6 40	.11715	297	.11688	295
1 20	.02328	291	.02328	291	6 50	.12012	297	.11983	295
1 30	.02619	291	.02619	291	7 00	.12309	298	.12278	296
1 40	.02910	291	.02910	291	7 10	.12607	298	.12574	295
1 50	.03201	292	.03201	291	7 20	.12905	298	.12869	296
2 00	.03493	291	.03492	291	7 30	.13203	299	.13165	296
2 10	.03784	292	.03783	292	7 40	.13502	299	.13461	297
2 20	.04076	291	.04075	291	7 50	.13801	299	.13758	296
2 30	.04367	292	.04366	292	8 00	.14100	300	.14054	297
2 40	.04659	292	.04658	291	8 10	.14400	300	.14351	297
2 50	.04951	292	.04949	292	8 20	.14700	301	.14648	297
3 00	.05243	292	.05241	292	8 30	.15001	300	.14945	298
3 10	.05535	293	.05533	291	8 40	.15301	302	.15243	297
3 20	.05828	292	.05824	292	8 50	.15603	301	.15540	298
3 30	.06120	293	.06116	292	9 00	.15904	303	.15838	299
3 40	.06413	292	.06408	292	9 10	.16207	302	.16137	298
3 50	.06705	293	.06700	293	9 20	.16509	303	.16435	299
4 00	.06998	293	.06993	292	9 30	.16812	303	.16734	299
4 10	.07291	294	.07285	293	9 40	.17115	304	.17033	300
4 20	.07585	293	.07578	292	9 50	.17419	305	.17333	300
4 30	.07878	294	.07870	293	10 00	.17724	304	.17633	300
4 40	.08172	294	.08163	293	10 10	.18028	306	.17933	300
4 50	.08466	294	.08456	293	10 20	.18334	305	.18233	301
5 00	.08760	294	.08749	293	10 30	.18639	307	.18534	301
5 10	.09054	295	.09042	293	10 40	.18946	306	.18835	301
5 20	.09349	295	.09335	294	10 50	.19252	308	.19136	302

TABLE IV.—CONTINUED.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
11°00′	.19560	308	.19438	302	16°30′	.30049	330	.29621	317
11 10	.19868	308	.19740	302	16 40	.30379	332	.29938	317
11 20	.20176	309	.20042	303	16 50	.30711	332	.30255	318
11 30	.20485	309	.20345	303	17 00	.31043	333	.30573	318
11 40	.20794	310	.20648	304	17 10	.31376	334	.30891	319
11 50	.21104	311	.20952	304	17 20	.31710	335	.31210	320
12 00	.21415	311	.21256	304	17 30	.32045	336	.31530	320
12 10	21726	311	.21560	304	17 40	.32381	336	.31850	321
12 20	.22037	313	.21864	305	17 50	.32717	338	.32171	321
12 30	.22350	313	.22169	306	18 00	.33055	339	.32492	322
12 40	.22663	313	.22475	306	18 10	.33394	339	.32814	322
12 50	.22976	314	.22781	306	18 20	.33733	341	.33136	324
13 00	.23290	315	.23087	306	18 30	.34074	341	.33460	323
13 10	.23605	315	.23393	307	18 40	.34415	343	.33783	325
13 20	.23920	317	.23700	308	18 50	.34758	344	.34108	325
13 30	.24237	316	.24008	308	19 00	.35102	344	.34433	325
13 40	.24553	318	.24316	308	19 10	.35446	346	.34758	327
13 50	.24871	318	.24624	309	19 20	.35792	347	.35085	327
14 00	.25189	319	.24933	309	19 30	.36139	347	.35412	328
14 10	.25508	319	.25242	310	19 40	.36486	349	.35740	328
14 20	.25827	320	.25552	310	19 50	.36835	350	.36068	329
14 30	.26147	321	.25862	310	20 00	.37185	351	.36397	330
14 40	.26468	322	.26172	311	20 10	.37536	353	.36727	330
14 50	.26790	322	.26483	312	20 20	.37889	353	.37057	331
15 00	.27112	323	.26795	312	20 30	.38242	355	.37388	332
15 10	.27435	324	.27107	312	20 40	.38597	355	.37720	333
15 20	.27759	325	.27419	313	20 50	.38952	357	.38053	333
15 30	.28084	325	.27732	314	21 00	.39309	358	.38386	335
15 40	.28409	327	.28046	314	21 10	.39667	360	.38721	334
15 50	.28736	327	.28360	315	21 20	.40027	360	.39055	336
16 00	.29063	328	.28675	315	21 30	.40387	362	.39391	336
16 10	.29391	328	.28990	315	21 40	.40749	363	.39727	338
16 20	.29719	330	.29305	316	21 50	.41112	364	.40065	338

TABLE IV.—Continued.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
22°00′	.41476	366	.40403	338	27°30′	.54320	418	.52057	370
22 10	.41842	367	.40741	340	27 40	.54738	420	.52427	371
22 20	.42209	368	.41081	340	27 50	.55158	422	.52798	373
22 30	.42577	370	.41421	342	28 00	.55580	423	.53171	374
22 40	.42947	371	.41763	342	28 10	.56003	426	.53545	375
22 50	.43318	372	.42105	342	28 20	.56429	427	.53920	376
23 00	.43690	373	.42447	344	28 30	.56856	430	.54296	377
23 10	.44063	375	.42791	345	28 40	.57286	431	.54673	378
23 20	.44438	377	.43136	345	28 50	.57717	434	.55051	380
23 30	.44815	378	.43481	347	29 00	.58151	436	.55431	381
23 40	.45193	379	.43828	347	29 10	.58587	438	.55812	382
23 50	.45572	381	.44175	348	29 20	.59025	440	.56194	383
24 00	.45953	382	.44523	349	29 30	.59465	442	.56577	385
24 10	.46335	384	.44872	350	29 40	.59907	445	.56962	386
24 20	.46719	385	.45222	351	29 50	.60352	447	.57348	387
24 30	.47104	387	.45573	351	30 00	.60799	449	.57735	389
24 40	.47491	388	.45924	353	30 10	.61248	451	.58124	389
24 50	.47879	390	.46277	354	30 20	.61699	453	.58513	391
25 00	.48269	392	.46631	354	30 30	.62152	456	.58904	393
25 10	.48661	393	.46985	356	30 40	.62608	459	.59297	394
25 20	.49054	395	.47341	357	30 50	.63067	460	.59691	395
25 30	.49449	396	.47698	357	31 00	.63527	463	.60086	397
25 40	.49845	398	.48055	359	31 10	.63990	466	.60483	398
25 50	.50243	400	.48414	359	31 20	.64456	468	.60881	396
26 00	.50643	402	.48773	361	31 30	.64924	471	.61280	401
26 10	.51045	403	.49134	361	31 40	.65395	473	.61681	402
26 20	.51448	405	.49495	363	31 50	.65868	475	.62083	404
26 30	.51853	407	.49858	364	32 00	.66343	479	.62487	405
26 40	.52260	408	.50222	365	32 10	.66822	480	.62892	407
26 50	.52668	410	.50587	366	32 20	.67302	484	.63299	408
27 00	.53078	413	.50953	367	32 30	.67786	486	.63707	410
27 10	.53491	414	.51320	368	32 40	.68272	489	.64117	411
27 20	.53905	415	.51688	369	32 50	.68761	492	.64528	413

TABLE IV.—CONTINUED.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
33°00′	.69253	494	.64941	414	38°30′	0.87275	609	.79544	476
33 10	.69747	498	.65355	416	38 40	0.87884	613	.80020	478
33 20	.70245	499	.65771	418	38 50	0.88497	617	.80498	480
33 30	.70744	504	.66189	419	39 00	0.89114	622	.80978	483
33 40	.71248	506	.66608	420	39 10	0.89736	627	.81461	485
33 50	.71754	509	.67028	423	39 20	0.90363	631	.81946	488
34 00	.72263	512	.67451	424	39 30	0.90994	635	.82434	489
34 10	.72775	515	.67875	426	39 40	0.91629	640	.82923	492
34 20	.73290	518	.68301	427	39 50	0.92269	645	.83415	495
34 30	.73808	522	.68728	429	40 00	0.92914	649	.83910	497
34 40	.74330	524	.69157	431	40 10	0.93563	655	.84407	499
34 50	.74854	528	.69588	433	40 20	0.94218	659	.84906	502
35 00	.75382	530	.70021	434	40 30	0.94877	664	.85408	504
35 10	.75912	535	.70455	436	40 40	0.95541	669	.85912	507
35 20	.76447	537	.70891	438	40 50	0.96210	674	.86419	510
35 30	.76984	541	.71329	440	41 00	0.96884	679	.86929	512
35 40	.77525	544	.71769	442	41 10	0.97563	685	.87441	514
35 50	.78069	548	.72211	443	41 20	0.98248	689	.87955	518
36 00	.78617	551	.72654	446	41 30	0.98937	696	.88473	519
36 10	.79168	554	.73100	447	41 40	0.99633	700	.88992	523
36 20	79722	558	.73547	449	41 50	1.00333	706	.89515	525
36 30	.80280	562	.73996	451	42 00	1.01039	712	.90040	529
36 40	.80842	566	.74447	453	42 10	1.01751	717	.90569	530
36 50	.81408	569	.74900	455	42 20	1.02468	723	.91099	534
37 00	.81977	573	.75355	457	42 30	1.03191	728	.91633	537
37 10	.82550	577	.75812	460	42 40	1.03919	735	.92170	539
37 20	.83127	580	.76272	461	42 50	1.04654	741	.92709	543
37 30	.83707	585	.76733	463	43 00	1.05395	746	.93252	545
37 40	.84292	588	.77196	465	43 10	1.06141	753	.93797	548
37 50	.84880	593	.77661	468	43 20	1.06894	759	.94345	551
38 00	.85473	596	.78129	469	43 30	1.07653	765	.94896	555
38 10	.86069	601	.78598	472	43 40	1.08418	772	.95451	557
38 20	.76670	605	.79070	474	43 50	1.09190	778	.96008	561

TABLE IV.—CONTINUED.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
44°00′	1.09968	785	0.96569	564	49°30′	1.40001	1067	1.17085	692
44 10	1.10753	792	0.97133	567	49 40	1.41068	1079	1.17777	697
44 20	1.11545	798	0.97700	570	49 50	1.42147	1089	1.18474	701
44 30	1.12343	805	0.98270	573	50 00	1.43236	1101	1.19175	707
44 40	1.13148	812	0.98843	577	50 10	1.44337	1113	1.19882	711
44 50	1.13960	819	0.99420	580	50 20	1.45450	1124	1.20593	717
45 00	1.14779	827	1.00000	583	50 30	1.46574	1136	1.21310	721
45 10	1.15606	833	1.00583	587	50 40	1.47710	1149	1.22031	727
45 20	1.16439	841	1.01170	591	50 50	1.48859	1161	1.22758	732
45 30	1.17280	849	1.01761	594	51 00	1.50020	1173	1.23490	737
45 40	1.18129	856	1.02355	597	51 10	1.51193	1186	1.24227	742
45 50	1.18985	864	1.02952	601	51 20	1.52379	1200	1.24969	748
46 00	1.19849	872	1.03553	605	51 30	1.53579	1212	1.25717	754
46 10	1.20721	879	1.04158	608	51 40	1.54791	1226	1.26471	759
46 20	1.21600	888	1.04766	612	51 50	1.56017	1240	1.27230	764
46 30	1.22488	896	1.05378	616	52 00	1.57257	1253	1.27994	770
46 40	1.23384	904	1.05994	619	52 10	1.58510	1268	1.28764	777
46 50	1.24288	913	1.06613	624	52 20	1.59778	1282	1.29541	782
47 00	1.25201	922	1.07237	627	52 30	1.61060	1297	1.30323	787
47 10	1.26123	930	1.07864	632	52 40	1.62357	1311	1.31110	794
47 20	1.27053	938	1.08496	635	52 50	1.63668	1327	1.31904	800
47 30	1.27991	948	1.09131	639	53 00	1.64995	1342	1.32704	807
47 40	1.28939	957	1.09770	644	53 10	1.66337	1359	1.33511	812
47 50	1.29896	967	1.10414	647	53 20	1.67696	1374	1.34323	819
48 00	1.30863	975	1.11061	652	53 30	1.69070	1390	1.35142	826
48 10	1.31838	985	1.11713	656	53 40	1.70460	1407	1.35968	832
48 20	1.32823	995	1.12369	660	53 50	1.71867	1424	1.36800	838
48 30	1.33818	1005	1.13029	665	54 00	1.73291	1441	1.37638	846
48 40	1.34823	1015	1.13694	669	54 10	1.74732	1459	1.38484	852
48 50	1.35838	1025	1.14363	674	54 20	1.76191	1476	1.39336	859
49 00	1.36863	1035	1.15037	678	54 30	1.77667	1495	1.40195	866
49 10	1.37898	1046	1.15715	683	54 40	1.79162	1513	1.41061	873
49 20	1.38944	1057	1.16398	687	54 50	1.80675	1532	1.41934	881

TABLE IV.—Continued.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
55°00′	1.82207	1551	1.42815	888	60°30′	2.46196	2455	1.76749	1206
55 10	1.83758	1571	1.43703	895	60 40	2.48651	2494	1.77955	1219
55 20	1.85329	1590	1.44598	903	60 50	2.51145	2533	1.79174	1231
55 30	1.86919	1611	1.45501	910	61 00	2.53678	2573	1.80405	1244
55 40	1.88530	1632	1.46411	919	61 10	2.56251	2614	1.81649	1257
55 50	1.90162	1653	1.47330	926	61 20	2.58865	2656	1.82906	1271
56 00	1.91815	1674	1.48256	934	61 30	2.61521	2699	1.84177	1285
56 10	1.93489	1697	1.49190	943	61 40	2.64220	2743	1.85462	1298
56 20	1.95186	1719	1.50133	951	61 50	2.66963	2789	1.86760	1313
56 30	1.96905	1741	1.51084	959	62 00	2.69752	2834	1.88073	1327
56 40	1.98646	1765	1.52043	967	62 10	2.72586	2882	1.89400	1341
56 50	2.00411	1788	1.53010	976	62 20	2.75468	2930	1.90741	1357
57 00	2.02199	1813	1.53986	986	62 30	2.78398	2980	1.92098	1372
57 10	2.04012	1837	1.54972	994	62 40	2.81378	3030	1.93470	1388
57 20	2.05849	1863	1.55966	1003	62 50	2.84408	3082	1.94858	1403
57 30	2.07712	1888	1.56969	1012	63 00	2.87490	3136	1.96261	1419
57 40	2.09600	1915	1.57981	1021	63 10	2.90626	3190	1.97680	1436
57 50	2.11515	1941	1.59002	1031	63 20	2.93816	3246	1.99116	1453
58 00	2.13456	1968	1.60033	1041	63 30	2.97062	3304	2.00569	1470
58 10	2.15424	1997	1.61074	1051	63 40	3.00366	3362	2.02039	1487
58 20	2.17421	2025	1.62125	1060	63 50	3.03728	3422	2.03526	1504
58 30	2.19446	2054	1.63185	1071	64 00	3.07150	3484	2.05030	1523
58 40	2.21500	2083	1.64256	1081	64 10	3.10634	3548	2.06553	1541
58 50	2.23583	2114	1.65337	1091	64 20	3.14182	3612	2.08094	1560
59 00	2.25697	2145	1.66428	1102	64 30	3.17794	3680	2.09654	1579
59 10	2.27842	2176	1.67530	1113	64 40	3.21474	3747	2.11233	1599
59 20	2.30018	2208	1.68643	1123	64 50	3.25221	3818	2.12832	1619
59 30	2.32226	2242	1.69766	1135	65 00	3.29039	3890	2.14451	1639
59 40	2.34468	2275	1.70901	1146	65 10	3.32929	3965	2.16090	1659
59 50	2.36743	2310	1.72047	1158	65 20	3.36894	4040	2.17749	1681
60 00	2.39053	2345	1.73205	1170	65 30	3.40934	4118	2.19430	1702
60 10	2.41398	2381	1.74375	1181	65 40	3.45052	4199	2.21132	1725
60 20	2.43779	2417	1.75556	1193	65 50	3.49251	4281	2.22857	1747

TABLE IV.—CONTINUED.

θ	(θ)	Diff	Tan θ	Diff	θ	(θ)	Diff	Tan θ	Diff
66°00′	3.53532	4366	2.24604	1770	68°00′	4.12255	5594	2.47509	2088
66 10	3.57898	4452	2.26374	1793	68 10	4.17849	5717	2.49597	2118
66 20	3.62350	4543	2.28167	1817	68 20	4.23566	5844	2.51715	2150
66 30	3.66893	4634	2.29984	1842	68 30	4.29410	5975	2.53865	2181
66 40	3.71527	4730	2.31826	1867	68 40	4.35385	6110	2.56046	2215
66 50	3.76257	4826	2.33693	1892	68 50	4.41495	6249	2.58261	2248
67 00	3.81083	4927	2.35585	1919	69 00	4.47744	6393	2.60509	2282
67 10	3.86010	5030	2.37504	1945	69 10	4.54137	6541	2.62791	2318
67 20	3.91040	5137	2.39449	1973	69 20	4.60678	6694	2.65109	2353
67 30	3.96177	5245	2.41422	2000	69 30	4.67372	6853	2.67462	2391
67 40	4.01422	5359	2.43422	2029	69 40	4.74225	7016	2.69853	2428
67 50	4.06781	5474	2.45451	2058	69 50	4.81241	7184	2.72281	2467
					70 00	4.88425	7359	2.74748	2506

TABLE V. FOR MORTAR-FIRING.

$\phi = 30°$. $Y_0 = 0.15X$.

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2243	142	9.13	30	31° 53′	7	274	7
310	2385	144	9.43	29	32 00	7	281	8
320	2529	146	9.72	28	32 07	7	289	7
330	2675	149	10.00	29	32 14	8	296	7
340	2824	153	10.29	29	32 22	7	303	7
350	2977	156	10.58	29	32 29	8	310	7
360	3133	159	10.87	29	32 37	8	317	7
370	3292	162	11.16	28	32 45	8	324	6
380	3454	163	11.44	28	32 53	8	330	7
390	3617	165	11.72	28	33 01	8	337	6
400	3782	167	12.00	28	33 09	9	343	6
410	3949	170	12.28	28	33 18	9	349	6
420	4119	172	12.56	28	33 27	9	355	6
430	4291	175	12.84	27	33 36	9	361	6
440	4466	177	13.11	28	33 45	8	367	6
450	4643	178	13.39	27	33 53	9	373	5
460	4821	180	13.66	27	34 02	9	378	6
470	5001	181	13.93	27	34 11	10	384	5
480	5182	183	14.20	27	34 21	9	389	5
490	5365	184	14.47	27	34 30	9	394	5
500	5549	187	14.74	27	34 39	10	399	5

$\phi = 35°$. $Y_0 = 0.19X$.

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2415	148	10.45	33	37° 08′	9	273	8
310	2563	152	10.78	33	37 17	9	281	7
320	2715	158	11.11	33	37 26	9	288	7
330	2873	162	11.44	33	37 35	9	295	7
340	3035	163	11.77	33	37 44	9	302	7
350	3198	164	12.10	32	37 53	9	309	7
360	3362	166	12.42	33	38 02	9	316	7
370	3528	171	12.75	32	38 11	9	323	6
380	3699	174	13.07	32	38 20	9	329	7
390	3873	175	13.39	32	38 29	10	336	6
400	4048	177	13.71	32	38 39	10	342	6
410	4225	180	14.03	32	38 49	10	348	6
420	4405	182	14.35	31	38 59	10	354	6
430	4587	182	14.66	31	39 09	10	360	6
440	4769	185	14.97	31	39 19	10	366	6
450	4954	186	15.28	31	39 29	10	372	5
460	5140	188	15.59	30	39 39	10	377	6
470	5328	190	15.89	31	39 49	11	383	5
480	5518	192	16.20	30	40 00	11	388	5
490	5710	194	16.50	30	40 11	11	393	5
500	5904	196	16.80	30	40 22	11	398	5

TABLE V.—(*Continued*).

$\phi = 40°. \quad Y_0 = 0.23X.$

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2514	155	11.69	37	42° 23′	9	273	7
310	2669	159	12.06	37	42 32	9	280	8
320	2828	163	12.43	37	42 41	9	288	7
330	2991	165	12.80	36	42 50	10	295	7
340	3156	168	13.16	36	43 00	9	302	7
350	3324	170	13.52	36	43 09	10	309	7
360	3494	172	13.88	36	43 19	10	316	6
370	3666	175	14.24	36	43 29	11	322	7
380	3841	177	14.60	36	43 40	10	329	6
390	4018	180	14.96	35	43 50	10	335	7
400	4198	184	15.31	36	44 00	10	342	6
410	4382	185	15.67	35	44 10	11	348	6
420	4567	185	16.02	34	44 21	11	354	6
430	4752	187	16.36	35	44 32	11	360	6
440	4939	188	16.71	34	44 43	11	366	6
450	5127	191	17.05	35	44 54	11	372	6
460	5318	193	17.40	34	45 05	11	378	5
470	5511	194	17.74	33	45 16	11	383	5
480	5705	194	18.07	33	45 27	11	388	5
490	5899	195	18.40	33	45 38	12	393	5
500	6094	197	18.73	32	45 50	11	398	5

$\phi = 45°. \quad Y_0 = 0.27X.$

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2541	157	12.83	41	47° 28′	9	273	8
310	2698	158	13.24	40	47 37	9	281	7
320	2856	160	13.64	41	47 46	10	288	8
330	3016	163	14.05	40	47 56	10	296	7
340	3179	169	14.45	39	48 06	10	303	7
350	3348	172	14.84	40	48 16	10	310	7
360	3520	173	15.24	39	48 26	10	317	7
370	3693	175	15.63	39	48 36	11	324	6
380	3868	178	16.02	39	48 47	10	330	7
390	4046	182	16.41	38	48 57	11	337	6
400	4228	182	16.79	39	49 08	11	343	7
410	4410	182	17.18	38	49 19	11	350	6
420	4592	184	17.56	38	49 30	11	356	6
430	4776	187	17.94	38	49 41	11	362	6
440	4963	188	18.32	37	49 52	11	368	6
450	5151	192	18.69	37	50 03	11	374	5
460	5343	192	19.06	37	50 14	11	379	6
470	5535	192	19.43	36	50 25	12	385	5
480	5727	194	19.79	37	50 37	11	390	6
490	5921	195	20.16	36	50 48	12	396	5
500	6116	196	20.52	36	51 00	12	401	5

TABLE V.—(*Continued*).

$\phi = 50°$. $Y_0 = 0.32X$.

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2499	150	13.89	43	52° 28′	9	275	8
310	2649	153	14.32	43	52 37	9	283	7
320	2802	157	14.75	43	52 46	9	290	7
330	2959	161	15.18	43	52 55	10	297	7
340	3120	164	15.61	43	53 05	9	304	7
350	3284	166	16.04	43	53 14	10	311	7
360	3450	167	16.47	43	53 24	9	318	7
370	3617	170	16.90	42	53 33	10	325	7
380	3787	173	17.32	41	53 43	10	332	7
390	3960	176	17.73	41	53 53	11	339	6
400	4136	178	18.14	41	54 04	11	345	7
410	4314	178	18.55	41	54 15	11	352	6
420	4492	179	18.96	41	54 26	10	358	7
430	4671	181	19.37	41	54 36	11	365	6
440	4852	183	19.78	40	54 47	10	371	6
450	5035	185	20.18	40	54 57	11	377	6
460	5220	186	20.58	40	55 08	11	383	6
470	5406	186	20.98	39	55 19	11	389	5
480	5592	187	21.37	39	55 30	11	394	6
490	5779	189	21.76	39	55 41	12	400	5
500	5968	189	22.15	39	55 53	11	405	6

$\phi = 55°$. $Y_0 = 0.38X$.

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2376	145	14.84	47	57° 21′	8	277	8
310	2521	147	15.31	46	57 29	9	285	7
320	2668	150	15.77	46	57 38	8	292	8
330	2818	153	16.23	45	57 46	9	300	7
340	2971	156	16.68	46	57 55	9	307	7
350	3127	158	17.14	45	58 04	9	314	7
360	3285	160	17.59	45	58 13	9	321	7
370	3445	162	18.04	44	58 22	9	328	7
380	3607	163	18.48	45	58 31	9	335	7
390	3770	165	18.93	44	58 40	10	342	6
400	3935	167	19.37	44	58 50	9	348	7
410	4102	170	19.81	44	58 59	10	355	6
420	4272	170	20.25	43	59 09	10	361	7
430	4442	170	20.68	43	59 19	10	368	6
440	4612	172	21.11	42	59 29	10	374	6
450	4784	175	21.53	43	59 39	10	380	6
460	4959	176	21.96	42	59 49	10	386	6
470	5135	176	22.38	42	59 59	11	392	6
480	5311	177	22.80	42	60 10	10	398	6
490	5488	180	23.22	41	60 20	11	404	6
500	5668	181	23.63	42	60 31	11	410	6

TABLE V.—(*Continued*).

$\phi = 60°$. $Y_0 = 0.47X$.

$\frac{V}{\sqrt{C}}$	$\frac{X}{C}$	D	$\frac{T}{\sqrt{C}}$	D	ω	D	$\frac{v_\omega}{\sqrt{C}}$	D
300	2189	133	15.67	49	62° 05′	8	277	8
310	2322	135	16.16	49	62 13	7	285	8
320	2457	138	16.65	49	62 20	8	293	8
330	2595	140	17.14	49	62 28	8	301	7
340	2735	143	17.63	48	63 36	8	308	8
350	2878	145	18.11	48	62 24	8	316	7
360	3023	148	18.59	48	62 52	9	323	7
370	3171	149	19.07	47	63 01	8	330	7
380	3320	152	19.54	46	63 09	9	337	7
390	3472	153	20.00	46	63 18	8	344	7
400	3625	154	20.46	46	63 26	9	351	7
410	3779	155	20.92	45	63 35	8	358	6
420	3934	156	21.37	46	63 43	9	364	7
430	4090	157	21.83	46	63 52	8	371	6
440	4247	158	22.29	45	64 00	9	377	7
450	4405	158	22.74	44	64 09	9	384	6
460	4563	159	23.18	44	64 18	9	390	6
470	4722	162	23.62	44	64 27	9	396	6
480	4884	163	24.06	44	64 36	9	402	6
490	5047	163	24.50	43	64 45	9	408	6
500	5210	164	24.93	43	64 54	9	414	6

TABLE VI.—SIACCI'S FACTORS (β).

Angle of Departure ϕ.	Range in Metres.									
	1000.	2000.	3000.	4000.	5000.	6000.	7000.	8000.	9000.	10000.
5°	1.00	1.00	1.00	..	..	..	..	..	..	..
6	1.00	0.99	0.99	..	..	..	..	..	..	..
7	1.00	0.99	0.98	..	..	..	..	..	..	..
8	1.00	9.99	0.96	..	..	..	..	..	..	..
9	1.01	1.00	0.97	..	..	..	..	..	..	..
10	1.01	1.01	0.98	0.98	..	..	..	..	..	..
11	1.01	1.01	0.99	0.98	..	..	..	..	..	..
12	1.01	1.01	0.99	0.98	..	..	..	..	..	..
13	1.01	1.01	1.00	0.98	0.95	..	..	..	..	..
14	1.01	1.01	1 00	0.98	0.95	..	..	..	..	..
15	1.01	1.01	1.00	0.98	0.95	..	..	..	..	..
16	1.02	1.02	1.01	0.98	0.96	0.93	..	..	..	..
17	1.02	1.02	1.01	0.98	0.96	0.93	..	..	..	..
18	1.02	1.02	1.02	0.98	0.96	0.93	0.91	..	..	..
19	1.02	1.02	1.02	0.99	0.97	0.93	0.91	..	..	..
20	1.03	1.03	1.02	0.99	0.97	0.93	0.90	..	..	..
21	1.03	1.03	1.02	1.00	0.97	0.93	0.90	..	..	..
22	1.03	1.03	1.02	1.00	0.98	0.94	0.90	..	..	..
23	1.04	1.03	1.03	1.01	0.98	0.94	0.90	..	..	..
24	1.04	1.04	1.03	1.01	0.99	0.94	0.90	..	..	..
25	1.04	1.04	1.03	1.02	0.99	0.94	0.90	0.87	0.84	..
26	1.05	1.05	1.04	1.02	1.00	0.95	0.91	0.07	0.84	..
27	1.05	1.05	1.04	1.03	1.01	0.95	0.91	0.87	0.84	..
28	1.05	1.05	1.05	1.03	1.01	0.96	0.91	0.88	0.84	..
29	1.06	1.06	1.05	1.04	1.02	0.96	0.92	0.88	0.84	..
30	1.06	1.06	1.06	1.05	1.02	0.97	0.92	0.88	0.84	0.80
31	1.07	1.07	1.06	1.05	1.02	0.97	0.93	0.88	0.84	0.80

TABLE VI.—(*Continued*).

Angle of Departure ϕ.	Range in Metres.									
	1000.	2000.	3000.	4000.	5000.	6000.	7000.	8000.	9000.	10000.
32°	1.07	1.07	1.07	1.05	1.03	0.98	0.93	0.88	0.84	0.80
33	1.08	1.08	1.07	1.06	1.04	0.99	0.94	0.88	0.84	0.80
34	1.09	1.09	1.08	1.06	1.04	0.99	0.95	0.89	0.84	0.80
35	1.09	1.09	1.08	1.06	1.04	1.00	0.95	0.89	0.84	0.80
36	1.10	1.09	1.08	1.07	1.05	1.01	0.96	0.89	0.84	0.80
37	1.11	1.10	1.09	1.08	1.06	1.03	0.96	0.90	0.85	0.80
38	1.11	1.10	1.09	1.08	1.06	1.04	0.97	0.91	0.85	0.80
39	1.12	1.11	1.10	1.09	1.07	1.05	0.98	0.91	0.85	0.80
40	1.13	1.12	1.11	1.10	1.08	1.06	0.99	0.92	0.85	0.80
41	1.14	1.13	1.12	1.10	1.08	1.06	0.99	0.92	0.86	0.80
42	1.14	1.14	1.13	1.11	1.09	1.07	1.00	0.93	0.86	0.80
43	1.15	1.15	1.14	1.12	1.10	1.08	1.01	0.93	0.86	0.80
44	1.16	1.16	1.15	1.13	1.11	1.09	1.02	0.94	0.87	0.81
45	1.18	1.17	1.16	1.14	1.12	1.10	1.03	0.95	0.87	0.81
46	1.19	1.18	1.17	1.15	1.13	1.11	1.03	0.95	0.87	0.81
47	1.20	1.19	1.18	1.17	1.15	1.12	1.04	0.96	0.88	0.81
48	1.21	1.21	1.20	1.18	1.16	1.13	1.05	0.96	0.88	0.81
49	1.23	1.22	1.21	1.20	1.18	1.14	1.05	0.97	0.88	0.81
50	1.24	1.23	1.22	1.21	1.19	1.15	1.06	0.97	0.89	0.81
51	1.25	1.24	1.23	1.22	1.20	1.16	1.07	0.98	0.89	0.81
52	1.27	1.26	1.25	1.24	1.22	1.18	1.08	0.98	0.89	0.81
53	1.29	1.28	1.27	1.26	1.23	1.19	1.09	0.99	0.89	0.81
54	1.30	1.29	1.28	1.27	1.25	1.20	1.10	0.99	0.90	0.82
55	1.32	1.31	1.30	1.29	1.26	1.21	1.11	1.00	0.90	0.82
56	1.34	1.33	1.32	1.31	1.28	1.23	1.12	1.00	0.90	0.82
57	1.37	1.36	1.35	1.34	1.29	1.24	1.12	1.00	0.90	0.82
58	1.39	1.38	1.37	1.36	1.31	1.25	1.13	1.00	0.90	0.81
59	1.42	1.41	1.40	1.38	1.33	1.26	1.13	1.00	0.90	0.81
60	1.45	1.44	1.43	1.40	1.35	1.27	1.14	1.01	0.90	0.81

TABLE FOR CONVERTING MILLIMETRES TO INCHES.

1 millimetre = 0.039370432 inches. Log = 8.5951702 − 10.

Millimetres.	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0394	0.0787	0.1181	0.1575	0.1969	0.2362	0.2756	0.3150	0.3543
1	0.3937	0.4331	0.4724	0.5118	0.5512	0.5906	0.6299	0.6693	0.7087	0.7480
2	0.7874	0.8268	0.8662	0.9055	0.9449	0.9843	1.0236	1.0630	1.1024	1.1418
3	1.1811	1.2205	1.2599	1.2992	1.3386	1.3780	1.4173	1.4567	1.4961	1.5355
4	1.5748	1.6142	1.6536	1.6929	1.7323	1.7717	1.8111	1.8504	1.8898	1.9292
5	1.9685	2.0079	2.0473	2.0867	2.1260	2.1654	2.2048	2.2441	2.2835	2.3229
6	2.3622	2.4016	2.4410	2.4804	2.5197	2.5691	2.5985	2.6378	2.6772	2.7166
7	2.7559	2.7953	2.8347	2.8741	2.9134	2.9528	2.9922	3.0316	3.0709	3.1103
8	3.1496	3.1890	3.2284	3.2678	3.3072	3.3565	3.3859	3.4253	3.4646	3.5040
9	3.5433	3.5828	3.6221	3.6615	3.7008	3.7402	3.7796	3.8190	3.8584	3.8977

Example.—The 17-centimetre German gun has a calibre of 172.6 millimetres, and a length of bore of 3784 millimetres. What are the equivalents in English inches?

170	millimetres	= 6.693	inches
2	"	= 0 079	"
0 6	"	= 0.024	"
172.6	"	= 6.796	"

3700	millimetres	= 145.67	inches
84	"	= 3.31	"
3784	"	= 148.98	"

TABLE FOR CONVERTING METRES TO FEET.

1 metre = 3.28086933 feet. Log = 0.5159890.

Metres.	Feet.	Metres.	Feet.	Metres.	Feet.
1	3.28	50	164.04	900	2952.78
2	6.56	60	196.85	1000	3280.87
3	9.84	70	229.66	2000	6561.74
4	13.12	80	262.47	3000	9842.61
5	16.40	90	295.28	4000	13123.48
6	19.69	100	328.09	5000	16404.35
7	22.97	200	656.17	6000	19685.22
8	26.25	300	984.26	7000	22966.09
9	29.53	400	1312.35	8000	26246.95
10	32.81	500	1640.43	9000	29527.82
20	65.62	600	1968.52	10000	32808.69
30	98.43	700	2296.61	20000	65617.39
40	131.23	800	2624.70	30000	98426 08

Example.—The mean range of the Krupp 12-cm siege gun, with an elevation of 5°, is 2894.3 metres. What is the range in feet?

2000	metres	=	6561.74	feet
800	"	=	2624.70	"
90	"	=	395.28	"
4	"	=	13.12	"
0.3	"	=	0.98	"
2894.3	"	=	9495.82	"

TABLE FOR CONVERTING METRE-TONNES TO FOOT-TONS.

1 metre-tonne = 3.2290518 foot-tons. Log = 0.5090750.

Metre-tonnes.	Foot-tons.	Metre-tonnes.	Foot-tons.	Metre-tonnes.	Foot-tons.
1	3.23	50	161.45	900	2906.15
2	6.46	60	193.74	1000	3229.05
3	9.69	70	226.03	2000	6458.10
4	12.92	80	258.32	3000	9687.16
5	16.15	90	290.61	4000	12916.21
6	19.37	100	322.91	5000	16145.26
7	22.60	200	645.81	6000	19374.31
8	25.83	300	968.72	7000	22603.36
9	29.06	400	1291.62	8000	25832.41
10	32.29	500	1614.53	9000	29061.47
20	64.58	600	1937.43	10000	32290.52
30	96.87	700	2260.34	20000	64581.04
40	129.16	800	2583.24	30000	96871.55

Example.—The muzzle energy developed by the Krupp 30.5-cm. gun is 6276 metre-tonnes. Express this in foot-tons.

6000	metre-tonnes		=	19374.31	foot-tons
200	"	"	=	645.81	"
70	"	"	=	226.03	"
6	"	"	=	19.37	"
6276	"	"	=	20265.52	"

TABLE FOR CONVERTING KILOGRAMMES TO POUNDS.

1 kilogramme = 2.20462132 pounds. Log = 0.3433340.

Kilo-grammes.	0	1	2	3	4	5	6	7	8	9
0	0.00	2.20	4.41	6.61	8.82	11.02	13.23	15.43	17.64	19.84
1	22.05	24.25	26.46	28.66	30.86	33.07	35.27	37.48	39.68	41.89
2	44.09	46.30	48.50	50.71	52.91	55.11	57.32	59.52	61.73	63.93
3	66.14	68.34	70.55	72.75	74.96	77.16	79.37	81.57	83.78	85.98
4	88.18	90.39	92.59	94.80	97.00	99.21	101.41	103.62	105.82	108.03
5	110.23	112.44	114.64	116.84	119.05	121.25	123.46	125.66	127.87	130.07
6	132.28	134.48	136.69	138.89	141.10	143.30	145.50	147.71	149.91	152.12
7	154.32	156.53	158.73	160.94	163.14	165.35	167.55	169.76	171.96	174.16
8	176.37	178.57	180.78	182.98	185.19	187.39	189.60	191.80	194.01	196.21
9	198.42	200.62	202.83	205.03	207.23	209.44	211.61	213.85	216.05	218.26
10	220.46	222.67	224.87	227.08	229.28	231.48	233.69	235.89	238.10	240 30
11	242.51	244.71	246.92	249.12	251.33	253.53	255.74	257.94	260.14	262 35
12	264.55	266.76	268.96	271.17	273.37	275.58	277.78	279.99	282.19	284.40
13	286.60	288.81	291.01	293.21	295.42	297.62	299.83	302 03	304.24	306.44
14	308.65	310.85	313.06	315.26	317.46	319.67	321.87	324.08	326.28	328.49
15	330.69	332.90	335.10	337.31	339.51	341.72	343.92	346.12	348.33	350.53
16	352.74	354.94	357.15	359.35	361.56	363.76	365.97	368.17	370.38	372.58
17	374.79	376.99	379.19	381.40	383.60	385.81	388.01	390.22	392.42	394.63
18	396.83	399.04	401.24	403.45	405.65	407.85	410.06	412.26	414.47	416.67
19	418.88	421.08	423.29	425.49	427.70	429.90	432.11	434.31	436.51	438.72

Example.—The 30.5-cm. Krupp gun fires a projectile weighing 455 kilogrammes. Express this in pounds.

450	kilogrammes	=	992	pounds
5	"	=	11	"
455	"	=	1003	"